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A DEGENERATION OF THE MODULI SPACE OF STABLE BUNDLES

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1. Let k be an algebraically closed field and let d be an odd integer. Let $g \geqslant 2$ be an integer and suppose Y is a smooth projective curve of genus g. Let U_Y be the set of isomorphism classes of stable bundles E of rank two and degree d. (Note: we do not fix $\wedge^2 E$.) Following Mumford and Seshadri, we know that U_Y is in a natural way the set of (closed) points of a smooth projective variety again denoted by U_Y. Our aim in this paper is to develop a method of studying the topology of U_Y by degeneration methods. Our main application is the proof of the following theorem conjectured by Newstead and Ramanan.

Theorem 1.1 ($k = \mathbf{C}$). *The kth Chern class of the tangent bundle of U_Y is zero in the deRham cohomology of U_Y if $k > 2g - 2$.*

We hope that degeneration methods may be useful in other contexts. For instance, one can hope that the theory can be generalized to bundles of arbitrary degree and rank. One should also be able to compute the lower Chern classes of $\Omega^1_{U_Y}$.

The following is a brief outline of this paper: Let X_0 be an irreducible curve of genus g which is smooth except of one ordinary node N. We let X be the normalization of X_0 and let P_1 and P_2 be the inverse image of N. Our object is to find a (singular) projective variety U_{X_0} which will play the role to U_Y when Y is smooth. In particular, if $\{Y_t\}$ is a family of smooth curves degenerating to X_0, then we desire that U_{Y_t} generates to U_{X_0}.

The first difficulty in constructing U_{X_0} is that one cannot hope that all the points of U_{X_0} will correspond to actual bundles on X_0. There are two methods to resolve this difficulty. One is to consider certain torsion-free sheaves on X_0 to obtain a candidate for U_{X_0} [3]. However, such a U_{X_0} does not appear to have (analytic) normal crossings. The second method, which we will follow, is to consider certain bundles on certain semistable models of X_0 as is suggested by

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the theory developed in [1]. The construction of U_{X_0} and the examination of its local properties occupy §§3 and 4. (§2 consists of definitions and elementary results on stable bundles on smooth curves.)

Let U be the set of all bundles E on X_0 so that the pullback $\tilde{E}$ of E to X is stable. We will see U is a smooth open subvariety of U_{X_0}. Letting $\tilde{U}_{X_0}$ be the normalization of U_{X_0}, we see $\tilde{U}_{X_0}$ is a smooth compactification of U. On the other hand, U is a fiber bundle with fiber GL(2) over S_0, the moduli space of stable bundles on X. The main object of the paper is to embed U into a certain projective fiber bundle S_3 over U_0 and then to obtain $\tilde{U}_{X_0}$ by blowing up and then blowing down S_3 in a fairly explicit way. The main result is Theorem 13.1. §§11 and 12 are independent of the rest of the paper. §11 gives conditions which insure a birational map between projective varieties is obtained by blowing up the target. §12 consists of a Chern class computation which together with Theorem 13.1 proves Theorem 1.1.

Since d is odd, we let $d = 2\alpha + 1$.

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2. We begin with some terminology. Let S be a scheme of finite type over k and let Z be a closed subscheme. Let E be a bundle on S.

Definition 1.1. A Z quotient of E is a locally free sheaf of $\mathcal{O}_Z$ modules Q and a surjection $\varphi: E \to Q$.

Two Z quotients $E \to Q$ and $E \to Q'$ are equal if $\operatorname{Ker}(E, Q) = \operatorname{Ker}(E, Q')$. We call $E' = \operatorname{Ker}(E, Q)$ the *modification* of E at Q. Let $F = \operatorname{Ker}(E_Z, Q)$. Then F is a sheaf of locally free $\mathcal{O}_Z$ modules and F is a Z quotient of E'. We call F the *canonical* Z quotient of the modification of E at Q. If $\mathcal{I}_Z$, the ideal sheaf of Z in S, is invertible, then E' is a bundle.

If $Z' \subseteq Z$, we call $Q \otimes \mathcal{O}_{Z'}$ the Z' quotient *induced* from Q. If Z'' is another closed subscheme of S, and Q'' is a Z'' quotient of E and $W \subseteq Z \cap Z''$, then we say Q and Q'' *coincide* over W if the induced W quotients are equal.

Remark 2.2. Suppose S is smooth, Z_1 and Z_2 are two divisors intersecting transversally, and Q is a Z_1 quotient of E. Let Q' be the induced $Z_1 \cap Z_2$ quotient of E_{Z_2}, $E' = \operatorname{Ker}(E, Q)$ and $E'' = \operatorname{Ker}(E_{Z_2}, Q')$. Then $(E')_{Z_2} = E''$, and the canonical quotients coincide over $Z_1 \cap Z_2$.

Now let C be a smooth projective curve, E a bundle of rank r over C and Q a locally free quotient of E.

Definition 2.3. Q is destabilizing (resp. semistabilizing) if

$$\frac{\deg Q}{\operatorname{rk} Q} < \frac{\deg E}{\operatorname{rk} E} \qquad \left(\text{resp. } \frac{\deg Q}{\operatorname{rk} Q} = \frac{\deg E}{\operatorname{rk} E}\right).$$

Now if C is rational, then E is a direct sum of line bundles L_i.

Definition 2.4. E is standard on a rational curve if deg L_i is either zero or one for each i.

If E is standard, there is a unique quotient Q so that Q is a direct sum of $\mathcal{O}_C$'s and $\mathrm{Ker}(E, Q)$ is a direct sum of $\mathcal{O}(1)$'s. We call Q the *standard quotient* of E.

X will be a fixed nonsingular curve of genus $g - 1$ for the rest of the paper and P_1 and P_2 will be fixed distinct points on X. Let E be a bundle of rank two on X and let L_i be P_i quotients of E of rank one. We suppose E is stable of odd degree $d = 2\alpha + 1$.

Lemma 2.5. (i) $E' = \mathrm{Ker}(E \to L_1)$ *is semistable. Further there is no semi-stabilizing quotient M of E' which coincides with the canonical P_1 quotient of E'.*

(ii) $E'' = \mathrm{Ker}(E \to L_1 \oplus L_2)$ *is stable unless there is an invertible quotient L of E of degree $\alpha + 1$ which coincides with L_1 and L_2.*

Proof. Consider case (i). Let M be a quotient of E' of degree $e \leqslant \alpha$ and let $M' = \mathrm{Ker}(E', M)$. If $e < \alpha$, then $\deg M' > \alpha$ so the subline bundle of E containing M' is destabilizing. If $e = \alpha$ and M coincides with the canonical quotient of E' at P_1, then the map from M' to E vanishes at P_1 and so $M'(P)$ maps to E. Then $M'(P)$ is destabilizing for E.

Consider case (ii). Then E'' has a quotient M of degree $e \leqslant \alpha - 1$ and hence a subbundle M' of degree $\geqslant \alpha$. But M' must be a subbundle of degree α of E, since otherwise E would have a subbundle of degree $> \alpha$. On the other hand, M' maps to zero in L_1 and L_2, so E/M' is a quotient of degree $\alpha + 1$ which coincides with L_{P_1} and L_{P_2}.

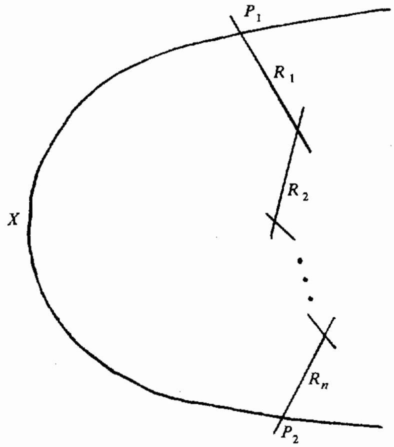
Lemma 2.6. *Suppose* $\deg E = 2\alpha + 1$ *and that E has a quotient Q of degree α. Further, suppose the L_i do not coincide with Q over P_i. Then*:

(i) $\mathrm{Ker}(E, L_1) = E'$ *is semistable. Further, Q is a quotient of E' and the canonical P_1 quotient of E' is glued to Q over P_1.*

(ii) $\mathrm{Ker}(E, L_1 \oplus L_2)$ *is stable unless there is a quotient Q' of E of degree $\alpha + 1$ which coincides with L_i over P_i.*

The proof of Lemma 2.6 is similar to that of Lemma 2.5.

3. Recall that X is a fixed nonsingular curve of genus $g - 1$. We let X_0 be the stable curve obtained by identifying P_1 with P_2. Now for each $n \geqslant 1$, we can define a semistable curve X_n whose components are X and nonsingular rational curves $R_1, \cdots, R_n$ with R_i meeting R_{i-1} and R_{i+1} and R_1 meeting P_1 and R_n meeting P_2.



Now we say a bundle $\tilde{E}$ of degree $2\alpha + 1$ on X is *slightly unstable* if it has a destabilizing quotient of degree α.

Now if E is a bundle of rank 2 on X_0, we let $\tilde{E}$ denote the pullback of E to X. If E is a bundle on X_i, we let $\tilde{E} = E \otimes \mathcal{O}_X$. We say E on X_i is weakly stable if it has no quotient line bundles Q which are destabilizing, i.e.,

$$2 \deg_{X_i} Q \leq \deg_{X_i} E.$$

We now consider certain types of bundles defined on X_0, X_1 and X_2. *We will assume all types are weakly stable of degree* $2\alpha + 1$.

Type I_s: E defined on X_0 and $\tilde{E}$ is stable.

Type I_u: E defined on X_0 and $\tilde{E}$ is slightly unstable.

Type II_1: E defined on X_1, $\tilde{E}$ is semistable, and E_{R_1} is standard of degree 1, i.e., $E_{R_1} \cong \mathcal{O}(1) \oplus \mathcal{O}$.

Type II_2: E is defined on X_1, $\tilde{E}$ is stable and E_{R_1} is standard of degree 2, i.e., $E_{R_1} \cong \mathcal{O}(1) \oplus \mathcal{O}(1)$.

Type III: E is defined on X_2, $\tilde{E}$ is stable, E_{R_i} is standard of degree 1, and the standard quotients of E_{R_1} and E_{R_2} do not coincide over $R_1 \cap R_2$.

Bundles of any of the above types will be called *potentially stable*. Now recall the setup of [1]. We let $d = 2\alpha + 1$ and $n = d + 2 - 2g$. We let W be a vector space of dimension n, G be the grassmannian of all codimension two subspaces of W and $\mathcal{E}$ be the universal bundle on G. We let $S_{g,d}$ be the Hilbert scheme of curves of degree d and genus g on G. We can then consider the stability of the m-Hilbert point of $C \in S_{g,d}$ for $m \gg 0$. Our first main result was that there was a d and an m so that if C is smooth, then C is m-Hilbert stable if and only if $\mathcal{E}_C$ is stable. Our second main result was that if C is m-Hilbert semistable, then C is semistable as a curve and $H^1(C, \mathcal{E}) = 0$. Further, if C is isomorphic to X_n, then $\mathcal{E}_C$ is potentially stable. We fix such a d and m.

Proposition 3.1. *Any potentially stable bundle on X_n is m-Hilbert semistable.*

Proof. Let U be the set of m-Hilbert stable points. First notice that if $\mathcal{E}_C$ is potentially stable, then $C \in \overline{U}$. Indeed, we can deform any semistable curve to a smooth curve, and we can then lift our bundle to this deformation. On the other hand, on a smooth curve any bundle can be deformed to a stable bundle. Indeed, we can write our bundle and a stable bundle as an extension of the same two line bundles. Since stability is an open condition, our claim follows.

Consequently, if $C \cong X_n$, we can find a smooth curve S, a family of semistable curves $\pi\colon Y \to S$, and a bundle F on Y so that for some $R \in S$, F_R on $Y_R = \pi^{-1}(R)$ is our potentially stable bundle and for $Q \neq R$, Y_Q is smooth and F_Q is stable. After replacing S by a cover possible ramified over R, we can assume there is a family $\pi'\colon Y' \to S$ and a bundle F' on Y' so that (Y', F') is isomorphic to (Y, F) over $S - R$ and so that F'_R is m-Hilbert semistable. Now let Y'' be the smooth surface which is the minimum model obtained by a resolution of singularities of Y. By uniqueness of minimum models, Y'' is also a minimum model of Y'. We will denote the pullbacks of F and F' to Y'' by F and F' again. We have an isomorphism ψ of F with F' over $S - R$. Now locally around R, we can choose a uniformizing parameter t of S at R and we can find a map $\varphi\colon F \to F'$ which is not identically zero on Y''_R. Indeed, we just take $\varphi = t^k\psi$ for some suitable k. Now let E and E' be F_R and F'_R on $Y''_R = Y_0$. Y_0 is isomorphic to X_n for some n. We let R be the union of the R_i's. We have nonzero maps φ and φ' from E to E' and from E' to E, and both E and E' are the pullbacks of potentially stable bundles. Our main claim is that if $\deg \tilde{E} \geqslant \deg \tilde{E}'$, then φ is an isomorphism. Our proof of this claim will only use the fact that E and E' are the pullbacks of potentially stable bundles, so Proposition 3.1 follows by reversing the roles of E and E' if necessary.

We first claim that if φ vanishes at P_1 and P_2, then φ vanishes on R. Indeed, we can write $E_R = L_1 \oplus L_2$ and $E'_R = M_1 \oplus M_2$ where each L_i and M_i has degree zero on all components of R except perhaps for one on which it has degree one. Now

$$\mathbf{Hom}_{\mathcal{O}_R}(E_R, E'_{R'}) = \bigoplus_{i,j} \left(L_i^{-1} \otimes M_j\right).$$

Our assumption means that the components φ_{ij} of φ vanish at P_1 and P_2. But $L_i^{-1} \otimes M_j$ has nonpositive degree on all components except perhaps for one R_k. Thus φ_{ij} vanish on all components except R_k, and hence vanish on R_k, since φ_{ij} vanishes at two points of R_k.

Second, we claim that if φ is an isomorphism at P_1 and P_2 and $\deg E_R = \deg E'_R$, then φ is an isomorphism over R. Consider $L = (\bigwedge^2 E_R)^{-1} \otimes \bigwedge^2 E'_R$. L has degree zero. First consider the case $\deg E'_R \leqslant 1$. Then L is either trivial on all components, or has degree one on some component R_i and degree -1 on

R_j. If L is trivial, then $\wedge^2 \varphi$ is nowhere zero and so φ is an isomorphism. Otherwise, $\wedge^2 \varphi$ vanishes on R_j and hence on all components above or below R_j depending on whether R_i is below or above R_j. So $\wedge^2 \varphi$ is zero at P_1 or P_2.

Next consider the case $\deg E'_R = 2$. The arguments given above eliminate all cases except if L has degree one on R_k and R_l and has degree zero on R_m if $m < k$ or $m > l$. In this case, let R' be the chain $R_{k+1} \cup \cdots \cup R_{l-1}$. Then $E_{R'}^{-1}$ has no sections since it is a direct sum $(L_1^{-1} \oplus L_2^{-1})_R$ where L_1 and L_2 both have degree one on some component of R'. Since $E'_{R'}$ is trivial, φ is identically zero on R'. Thus $\wedge^2\varphi$ vanishes to order two at $R_k \cap R_{k+1}$ and hence vanishes on R_k. Thus $\wedge^2\varphi$ vanishes at P_1.

Third, we claim that if $\deg E'_R \leqslant 1$ and φ has rank one at P_1 and P_2, then there is quotient line bundle L of E'_R so that $\varphi(E_R)$ maps to zero in L. Indeed, $\wedge^2\varphi$ must be zero since $\wedge^2 E'_R$ has positive degree on at most one component R_l and its degree on that component is one. Since $\wedge^2\varphi$ vanishes at P_1 and P_2, we see $\wedge^2\varphi = 0$. Now φ factors through a subline bundle of E'_R unless φ vanishes at some node $R_i \cap R_{i+1}$. But then the components φ_{pq} vanish at this node. We may assume $l \leqslant i$. Then E'_{R_j} is trivial for $j \geqslant i$ so φ_{pq} vanishes at P_2 and hence φ vanishes at P_2. This contradicts our assumption.

Fourthly, if $\tilde{E}$ and $\tilde{E}'$ have the same degree, we claim $\tilde{\varphi}$ cannot vanish at P_1 and P_2. Suppose not. Our claim is clear if $\tilde{E}$ and $\tilde{E}'$ are semistable. We have a nonzero homomorphism $\tilde{\varphi}$ from $\tilde{E}$ to $\tilde{E}'(-P_1 - P_2)$. Now $\wedge^2\tilde{\varphi}$ vanishes, since $\deg \tilde{E} > \deg \tilde{E}'(-P_1 - P_2)$. Hence φ factors through a subline bundle L of $\tilde{E}'(-P_1 - P_2)$. But L must have degree at least α, since $\tilde{E}$ is at most slightly unstable. But $\deg L \leqslant \alpha - 1$, since $\tilde{E}'$ is at most slightly unstable. Hence $\tilde{\varphi}$ is zero and hence φ is zero by the first claim.

Fifthly, we claim that if $\deg \tilde{E} > \deg \tilde{E}'$ then $\tilde{E}$ is slightly unstable and $\tilde{E}'$ is strictly semistable. Further, $\tilde{\varphi}$ has rank one at P_1 and P_2. In fact, $\tilde{\varphi}$ must factor through the destabilizing quotient of $\tilde{E}$, which must be a subbundle of $\tilde{E}'$.

We lastly claim φ is an isomorphism. Indeed suppose first that $\deg \tilde{E} > \deg \tilde{E}'$. Then from our fifth claim we see $\deg E_R = 0$ and $\deg E'_R = 1$. Our third claim shows there is quotient line bundle of E'_R which coincides with $\operatorname{coker} \tilde{\varphi}$ over P_1 and P_2. Hence there is a quotient line bundle L of E' so that $\varphi(E)$ maps to zero in L. Let $M = \operatorname{Ker}(E', L)$. Then $\deg M \leqslant \alpha$, since E' is weakly stable. But the map of $\tilde{E}$ to $\tilde{M}$ factors through the destabilizing quotient of $\tilde{E}$, so $\deg \tilde{M} > \alpha$. Since φ_R has rank one on all the $R_i \cap R_{i+1}$ and $\tilde{E}_R = \mathcal{O}_R \oplus \mathcal{O}_R$, we see $\deg \tilde{M}_R \geqslant 0$. Hence $\deg \tilde{M}_R = 0$ and $\deg \tilde{M} = \alpha$. Further M is a quotient of E, so $\deg M \geqslant \alpha + 1$ since E is weakly stable.

Thus we may assume $\deg \tilde{E} = \deg \tilde{E}'$. Now $\wedge^2\tilde{\varphi} = 0$, since if $\wedge^2\tilde{\varphi} \neq 0$, then $\tilde{\varphi}$ is an isomorphism since $\deg \tilde{E} = \deg \tilde{E}'$. Hence φ would be an isomorphism. Now if $\wedge^2\tilde{\varphi} = 0$, either $\tilde{E}$ or $\tilde{E}'$ must be unstable, since any nonzero map between two stable bundles of the same degree and rank is an isomorphism. Hence $\deg E'_R \leqslant 1$. If $\deg E'_R = 1$, then $\tilde{E}$ and $\tilde{E}'$ are semistable and $\tilde{\varphi}$ has rank one at P_1 and P_2. One sees as above that φ factors through a subbundle M of E' which is a quotient of E. This contradicts the weak stability of E and E'. Finally, suppose $\deg E'_R = 0$. If φ_R vanished at P_1, it would also vanish at P_2 and vice versa, since E_R and E'_R are trivial. Hence φ has rank one at P_1 and P_2. This again contradicts the weak stability of E and E'.

Corollary of proof 3.2. If E is potentially stable on X_n, then $\mathrm{Hom}(E, E) = k$.

4. Let C be a smooth curve and let $tP \in C$. Let π: $\mathcal{X} \to C$ be a flat family of curves with $\mathcal{X}$ smooth over k and $\pi^{-1}(P) = \mathcal{X}_P = X_0$, where X_0 is our nodal curve. We assume π has a section and that π is smooth away from P. Now let $F(T)$ be the set of closed subschemes $Y \subseteq (\mathcal{X} \times_C T) \times_k G$ which have the following properties:

(i) Y is a flat family of curves of genus g over T.

(ii) The induced map from Y to $T \times_k G$ is an embedding and for each closed $t \in T$, the corresponding curve in G is Hilbert semistable.

(iii) Locally on T, there is an isomorphism between the relative dualizing sheaf $\omega_{Y/T}$ and the pullback of $\omega_{\mathcal{X}/C}$.

We will see (iii) is an open condition. Assuming this, we see F is an open subfunctor of the relative Hilbert scheme of $\mathcal{X} \times G$, and so F is representable by a scheme p: $\mathcal{Y} \to C$. First notice that a closed point of $\mathcal{Y}$ lying over P consists of a curve $X' \subseteq G$ which is m-Hilbert semistable and a map φ of X' to X_0 so that the pullback of ω_X is $\omega_{X'}$. Since the genus of X_0 and X' are the same, one sees that X' is X_n, $0 \leqslant n \leqslant 2$, and the map from X' to X_0 is the map which collapses the R_i to the unique node of X_0.

Proposition 4.1. *$\mathcal{Y}$ is smooth over k and $\mathcal{Y}_P$ is a reduced divisor with normal crossings. Further $SL(W)$ operates freely on $\mathcal{Y}$.*

Before proving the proposition, we need to study deformations of X_n. Let A be an artinian k algebra. Then Schessingers's theory shows that given a flat deformation Z of X_n over $\operatorname{Spec} A$, there are $a_1, \cdots, a_n \in A$ so that at the ith node $N_i = R_i \cap R_{i+1}$, we have $\hat{\mathcal{O}}_{Z,N_i} \cong A[[x, y]]/(xy - a_i)$. The a_i are determined up to a unit, so we refer to $a_i = 0$ as the equation of the ith node. If Z' is another deformation of X_n so that a_i is the equation of the ith node of Z', then Z and Z' are locally isomorphic in the Zariski topology.

Now let $W = \operatorname{Spec} k[[t_1, \cdots, t_n]]$. We choose t to be a uniformizing parameter in $\mathcal{O}_{C,P}$, and we map W to C so that $t = t_1 \cdots t_n$.

Lemma 4.2. *There is a deformation Z of X_n over W and a morphism ψ: $Z \to \mathcal{X} \times_C W$ so that $\psi^*(\omega_{\mathcal{X}/C}) \cong \omega_{Z/W}$ and so that $t_i = 0$ is the equation for the ith node of X_n.*

Proof. Consider the case $n = 1$. There is an étale cover $q\colon Y_0 \to X_0$ so that Y_0 consists of two copies of X with P_1 on copy one glued to P_2 on copy two and vice versa. Let $\hat{\mathfrak{X}}$ be the completion of $\mathfrak{X}$ at X_0. Then we can find an étale cover $q\colon \hat{Y} \to \hat{X}$ reducing to $Y_0 \to X_0$ over P. Now let Q_1 and Q_2 be the two nodes of Y_0. We can find local parameters x and y at Q_1 so $t = xy$. Thus in some neighborhood of $Q_1 \in \hat{Y} \times_{\hat{C}} W$, we have $xy = t_1 t_2$. Let $\mathcal{I}_1$ be the ideal generated by x and t_1. One checks $\mathcal{I}_1$ is a Cartier divisor away from Q_1. Let $\mathcal{I}_2 = i^*(\mathcal{I}_1)$, where i is the involution on $\hat{Y}$. Now blow up $\hat{Y}$ at $\mathcal{I}_1$ around Q_1 and $\mathcal{I}_2$ around Q_2 to obtain $\hat{Y}'$. Since $\mathcal{I}_1$ and $\mathcal{I}_2$ are Cartier away from Q_1 and Q_2, this operation is well defined. Now i still acts on $\hat{Y}'$ as a fixed point free automorphism, so we can form the quotient Z by dividing $\hat{Y}'$ by the action of i.

We can check the local behavior of $\hat{Y}'$. Let

$$R = k[[x, y, t_1, t_2]]/(xy - t_1 t_2)$$

and consider $R_1 = R[x/t_1]$ and $R_2 = R[t_1/x]$. Let m be a maximal ideal in R_1. Let a be the residue class of x in R_1/m and let $x' = x/t_1$. If $a \neq 0$, then $x' - a$, t_1 and t_2 generate m since $y = t_2/x'$. Thus R_1 is regular at such a point and the equations $t_1 = t_2 = 0$ define a smooth curve. If $x' \in m$, then we have the relation $t_2 = x'y$, so x', y and t_1 generate m. Thus R_1 is regular at such a point and t_2 is the equation for the node. Now since $t_1/x = y/t_2$ we see $R_2 = R[y/t_2]$ and so t_1 is the equation for the node $y/t_2 = x = t_1 = 0$. So the two nodes of Z have equation $t_1 = 0$ and $t_2 = 0$. Similarly, one can check locally $\psi^*(\omega_{\mathfrak{X}/C}) = \omega_{Z/W}$.

The case of $n > 1$ is handled similarly.

Now let $\pi\colon X_n \to X_0$ be the standard map. There is a map from $\pi^*\Omega^1_{X_0}$ to $\Omega^1_{X_n}$ and hence a map

$$\varphi\colon \mathbf{Hom}\left(\Omega^1_{X_n}, \mathcal{O}\right) \to \mathbf{Hom}\left(\pi^*\Omega^1_{X_0}, \mathcal{O}\right).$$

Let E be the union of the R_i's in X_n.

Lemma 4.3. $\operatorname{Ker}\varphi \cong \oplus\, \mathcal{O}_{R_i}$, $\operatorname{Coker}\varphi \subseteq \mathcal{O}_E(-P_1) \oplus \mathcal{O}_E(-P_2)$.

Proof. We have a (nonexact) sequence of maps over some étale neighborhood of E,

$$\mathcal{O} \oplus \mathcal{O} \xrightarrow{\varphi_3} \pi^*\Omega^1_{X_0} \to \Omega^1_{X_n} \to \Omega^1_E. \tag{4.3.1}$$

To define φ_3, we let the node of X_0 be given by $xy = 0$. Then $\varphi_3(f, g) = f\,dx + g\,dy$. Dualizing φ_3, we obtain a map

$$\psi_3'\colon \mathbf{Hom}(\pi^*\Omega^1, \mathcal{O}) \to \mathcal{O} \oplus \mathcal{O}.$$

Let

$$\psi_3''\colon \mathcal{O} \oplus \mathcal{O} \to \mathcal{O}_E \oplus \mathcal{O}_E$$

be the natural map and let $\psi_3 = \psi_3'' \circ \psi_3'$. Then dualizing (4.3.1), we have

$$(4.3.2)\quad 0 \to \mathbf{Hom}\left(\Omega_E^1, \mathcal{O}\right) \xrightarrow{\psi_1} \mathbf{Hom}\left(\Omega_{X_n}^1, \mathcal{O}\right) \xrightarrow{\psi_2} \mathbf{Hom}\left(\pi^*\Omega_{X_0}^1, \mathcal{O}\right) \xrightarrow{\psi_3} \mathcal{O}_E \oplus \mathcal{O}_E.$$

We claim (4.3.2) is exact and that $\operatorname{im}\psi_3 \subseteq \mathcal{O}_E(-P_1) \oplus \mathcal{O}_E(-P_2)$. Our claim is readily verified except at P_1 and P_2. So consider $P_1 \in X_n$. We can introduce a local parameter y' at P_1 so that X_n is defined by $xy' = 0$ near P_1. Further, $y = 0$ near P_1. Now let $e_1 = \pi^*(dx)$ and $e_2 = \pi^*(dy)$. Near P_1, $\pi^*\Omega_{X_0}^1$ is generated by e_1 and e_2 subject to $xe_2 = 0$. Near P_1, $\Omega_{X_n}^1$ is generated by e_1 and dy' subject to $xdy' + y'e_1 = 0$. Further, e_2 maps to zero in $\Omega_{X_n}^1$ at P_1. Let us check the image of ψ_3 is contained in $\mathcal{O}(-P_1) \oplus \mathcal{O}(-P_2)$. Let $\Lambda \in \mathbf{Hom}(\pi^*\Omega_{X_0}^1, \mathcal{O})$ be defined near P_1. Then $x\Lambda(dy) = 0$, so $\Lambda(dy)$ must vanish on the curve $y' = 0$. Hence $\Lambda(dy)$ vanishes at P_1. Thus the image of ψ_3 is contained in $\mathcal{O}_E(-P_1) \oplus \mathcal{O}_E$ near P_1. Near P_2, the image is contained in $\mathcal{O}_E \oplus \mathcal{O}_E(-P_2)$. Our other assertions may be similarly verified. Further, $\operatorname{Hom}(\Omega_E^1, \mathcal{O}) = \bigoplus \mathcal{O}_{R_i}$.

Corollary 4.4. *The natural map*

$$\psi\colon H^1\left(\mathbf{Hom}\left(\Omega_{X_n}^1, \mathcal{O}\right)\right) \to H^1\left(\mathbf{Hom}\left(\pi^*\left(\Omega_{X_0}^1\right), \mathcal{O}\right)\right)$$

is injective.

Proof. $H^1(\bigoplus \mathcal{O}_{R_i}) = 0$ and $H^0(\mathcal{O}_E(-P_1) \oplus \mathcal{O}_E(-P_2)) = 0$.

Let A be an artin local k algebra, let $T = \operatorname{Spec} A$ and suppose T is a C scheme with the closed point of T going to R. Let t be a uniformizing parameter in $\mathcal{O}_{C,R}$.

Proposition 4.5. *Suppose Z' is a flat deformation of X_n over T and that there is a T morphism ψ' of Z' to $\mathcal{X} \times_C T$ which reduces to the standard morphism of X_n to X_0. Then there is a map of T to W so that Z' and ψ' are isomorphic to the pullbacks of the Z and ψ of Lemma* 4.2. *Further, if $a_i = 0$ is the equation of the ith node of Z', then $\prod_i a_i$ generates the ideal (t) in A.*

Proof. We may assume that A has a principal ideal (ε) of k-dimension 1 and that the proposition is true for $A_0 = A/(\varepsilon)$. Let $T_0 = \operatorname{Spec} A_0$.

We can find a map φ_1 of T to W so that the pullback (Z_1'', ψ_1'') of (Z, ψ) is isomorphic to (Z', ψ') over T_0 and so that the equations for the ith nodes of Z_1'' and Z' agree. Thus Z' and Z_1'' are locally isomorphic over T. Choose an affine open cover U_i of X_n and let U_i' and U_i'' be the corresponding open covers of Z' and Z''. We may assume there are isomorphisms $\Phi_i\colon \mathcal{O}_{U_i'} \to \mathcal{O}_{U_i''}$ which agree over $T_0 = \operatorname{Spec} A_0$. Now on $U_i' \cap U_j'$, we have

$$\Phi_i(f) = \Phi_j(f) + \varepsilon\lambda_{ij}(df)$$

for all $f \in \Gamma(U_i', \mathcal{O})$ and some $\lambda_{ij} \in \Gamma(U_i \cap U_j, \mathbf{Hom}(\Omega_{X_n}^1, \mathcal{O}))$. As usual, λ_{ij} is a cocycle and Z' and Z_1'' are isomorphic over C if and only if $\{\lambda_{ij}\}$ is a coboundary. Let μ_{ij} be the image of λ_{ij} in $\Gamma(U_i \cap U_j, \mathbf{Hom}(\pi^*\Omega^1_{X_0}, \mathcal{O}))$, and let μ be the corresponding cohomology class in $H^1(\mathbf{Hom}(\pi^*\Omega^1_{X_0}, \mathcal{O}))$. But μ is just the obstruction to extending the map from Z_0' to $\mathfrak{X} \times_C T_0$ to a C morphism from Z' to $\mathfrak{X} \times_C T$. Corollary 4.5 shows λ_{ij} is a coboundary.

Next we study the action of $\mathrm{SL}(W)$ on the Hilbert scheme of G.

Lemma 4.6. *Suppose λ: $X_n \to G$ is an embedding with $n \geq 1$ and $\lambda(X_n)$ is a potentially stable curve. Suppose $\sigma \in SL(W)$ fixes $\lambda(X_n)$ and is the identity on $X \subseteq X_n$. Then σ is a multiple of the identity.*

Proof. Let $E = \lambda^*(\mathcal{E})$. The map λ is given by choosing a basis $\{s_i\}$ of $H^0(X_n, E)$ and $\sigma \circ \lambda$ is obtained by a basis $\{s_i'\}$ of $H^0(X_n, E)$. Since σ induces the identity on X, we have that $s_i = s_i'$ on X up to a scalar multiple. But $H^0(X_n, E) \to H^0(X, E)$ is injective since E is potentially stable. Further, since $W = H^0(X, E)$, we see σ is the identity up to a scalar multiple.

With the same notation as in Lemma 4.6, we have

Lemma 4.7. *Let D be a vector field on G which is tangent to $\lambda(X_n)$. Then D is zero.*

Proof. Let $k[\varepsilon] = k[x]/(x^2)$. If V is a scheme over k, we let $V[\varepsilon] = V \times_k \operatorname{Spec} k[\varepsilon]$. Now D induces a $k[\varepsilon]$ map from $G[\varepsilon]$ to $G[\varepsilon]$. Note that D is zero on $\lambda(P_1)$ and $\lambda(P_2)$ since D is tangent to $\lambda(X_n)$. Further, D actually induces a map of $\lambda(X_n)[\varepsilon]$, and hence we may regard D: $X_n[\varepsilon] \to X_n[\varepsilon]$. Now there are no vector fields on X vanishing at P_1 and P_2, so D vanishes on $X \subseteq X_n$. Now the map from $G[\varepsilon]$ to $G[\varepsilon]$ is just given by a new basis $S_1', \cdots, S_n'$ of $H^0(G, \mathcal{E})[\varepsilon]$ which coincides with the old basis $S_1, \cdots, S_n \bmod \varepsilon$. Since D is zero on $X \subseteq X_n$, we have that $S_i = S_i'$ on $X[\varepsilon]$ up to a scalar multiple. We may assume $S_i = S_i'$ on $X[\varepsilon]$. But the map from $H^0(X_n[\varepsilon], \mathcal{E})$ to $H^0(X[\varepsilon], \mathcal{E})$ is injective, so the $S_i = S_i'$ in $H^0(X_n[\varepsilon], \mathcal{E})$. Hence $S_i = S_i'$ in $H^0(G[\varepsilon], \mathcal{E})$ and so $D = 0$.

With the notation of Lemma 4.6 again, we have

Lemma 4.8. *Let $\mathcal{I}$ be the ideal sheaf of $\lambda(X_n)$. Then the natural map*

$$p: H^0(G, T_G) \to \operatorname{Hom}\left(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_{\lambda(X_n)}\right)$$

is injective.

Proof. Identify $\lambda(X_n)$ and X_n. First note that the map from $H^0(G, T_G)$ to $H^0(X_n, T_G \otimes \mathcal{O}_{X_n})$ is injective. Indeed on G, we have a tautological exact sequence

$$0 \to \mathcal{J} \to \mathcal{O}^r \to \mathcal{E} \to 0,$$

where we denote $\dim W$ by r instead of by n as above. Now $T_G = \mathbf{Hom}(\mathcal{J}, \mathcal{E})$, so we have an exact sequence

$$0 \to \mathbf{Hom}(\mathcal{E}, \mathcal{E}) \to \mathcal{E}^r \to T_G \to 0.$$

Thus we have the commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^0(G,\mathbf{Hom}(\mathcal{E},\mathcal{E})) & \longrightarrow & H^0(G,\mathcal{E})^r & \longrightarrow & H^0(G,T_G) & \longrightarrow & 0 \\ & & \downarrow\alpha & & \downarrow\beta & & \downarrow\gamma & & \\ 0 & \longrightarrow & H^0(X_n,\mathbf{Hom}(\mathcal{E},\mathcal{E})_{X_n}) & \longrightarrow & H^0(X_n,\mathcal{E}_{X_n})^r & \longrightarrow & H^0(X_n,(T_G)_{X_n}) & & \end{array}$$

Now α is an isomorphism, since $\mathrm{Hom}(\mathcal{E}_{X_n},\mathcal{E}_{X_n})=k$. Further, β is an isomorphism. So γ is injective. Lemma 4.8 shows $(\mathrm{im}\,\gamma)\cap H^0(X,T_X)=0$. On the other hand, from the exact sequence

$$0\to \mathcal{I}/\mathcal{I}^2\to\left(\Omega^1_E\right)_{X_n}\to\Omega^1_{X_n}\to 0$$

we see that the following is exact:

$$0\to H^0\left(X_n,T_{X_n}\right)\to H^0\left(X_n,T_G\right)\to \mathrm{Hom}\left(\mathcal{I}/\mathcal{I}^2,\mathcal{O}_{X_n}\right).$$

So our claim follows.

Proof of Proposition 4.1. Suppose A is an artinian local k algebra with an ideal (ε) of dimension one. Let $A_0=A/(\varepsilon)$, $T=\mathrm{Spec}\,A$ and $T_0=\mathrm{Spec}\,A_0$. To show $\mathcal{Y}$ is smooth, it suffices to show that $F(T)$ surjects to $F(T_0)$. Let $Y_0\subseteq(\mathcal{X}\times_X T_0)\times_k G$ be an element of $F(A_0)$. Consider Y_0 as an abstract deformation of X_n mapping to $\mathcal{X}\times_C T_0$. Then by Proposition 4.5 there is an extension Y of Y_0 over T and a map of T to C so that Y maps to $\mathcal{X}\times_C T$. Further, $\mathcal{E}_0=\mathcal{E}_{Y_0}$ can be extended to a bundle $\mathcal{F}$ on Y, since the obstruction to lifting $\mathcal{E}_0$ lies in $H^2(\mathbf{Hom}(\mathcal{E},\mathcal{E})_{X_n})$. Finally, the sections of $\mathcal{E}_0$ defining the given map of Y_0 to G can be extended to $\mathcal{F}$, since $H^1(X_n,\mathcal{E}_{X_n})=0$. Thus Y is smooth over k.

Consider $\mathcal{Y}_P$ as a Cartier divisor defined by $t=0$. Let R be a closed point of $\mathcal{Y}_P$. Now over $T=\mathrm{Spec}\,\hat{\mathcal{O}}_{\mathcal{Y},R}$ there is a universal family of curves $Y\subseteq(\mathcal{X}\times_C T)\times_k G$. Let $z_i\in\hat{\mathcal{O}}_{\mathcal{Y},R}$ be the equation for the ith node of Y. Then $\prod z_i$ generates t (Proposition 4.5). On the other hand, for each i if $T_0=\mathrm{Spec}\,k[\varepsilon]/(\varepsilon^2)$, we can find $Y_i\in F(T_0)$ mapping to R, so that if a_{ij} is the equation for the jth node on Y_i, then $a_{ij}=\delta_{ij}$. Hence the dz_i are independent, and $\mathcal{Y}_P$ has normal crossings on $\mathcal{Y}$.

Let $S_{g,d}$ be the Hilbert scheme of curves of genus g and degree d in G and let $U\subseteq S_{g,d}$ be the set of semistable points. Since $g\geqslant 2$, there are at most finitely many maps σ of a semistable curve to its stable model. Thus the induced map

$$\varphi\colon\mathcal{Y}\to U\times_k C$$

is quasifinite. We claim φ is finite. Suppose not. Then we can find a smooth curve S, a $R\in S$ and a map $\psi\colon S-R\to\mathcal{Y}$ which induces a map ψ' of S to $U\times C$. We consider first the case when $S-R$ maps to $C-P$ and R maps to P. Using an argument similar to that used in Proposition 3.1, we see that the family of semistable curves induced by ψ' over S maps to the family of stable curves $\mathfrak{X} \times_C S$. But then ψ can be defined at R. The other cases can be similarly handled, so φ is finite.

$\mathrm{SL}(W)$ acts on $U \times C$ and on $\mathfrak{Y}$ over k. We can cover U by invariant affines U_i on which $\mathrm{SL}(W)$ acts properly and so cover Y by invariant affines $V_i = \varphi^{-1}(U_i \times C)$ on which $\mathrm{SL}(W)$, acts properly. We claim $\mathrm{SL}(W)$ acts freely. Lemma 4.6 shows that $\mathrm{SL}(W)$ has no fixed points. Indeed, if $\mathrm{SL}(W)$ fixed a closed point of $\mathfrak{Y}$, it would induce a nontrivial automorphism of $X \subseteq X_n$. But then if p is a standard map of X_n to X_0, we would have $\sigma \circ p \neq p$. Thus σ has no fixed points. Further, Lemma 4.8 shows the map of the Lie algebra of $\mathrm{SL}(W)$ to the tangent space of the Hilbert scheme at a given closed point is injective. So our claim is established.

So a geometric quotient $\mathfrak{W}$ of $\mathfrak{Y}$ by $\mathrm{SL}(W)$ exists [3, Proposition 3.12]. Further, $\mathfrak{Y}$ is a principal $\mathrm{SL}(W)$ bundle over $\mathfrak{W}$ [1, Proposition 0.9]. Since $\mathfrak{Y}$ is smooth, $\mathfrak{W}$ is also smooth. Further, $\mathfrak{W}_P$ has normal crossings on $\mathfrak{W}$. We finally claim that $\mathfrak{W}$ is projective over C. If $\mathfrak{W}$ is not proper over C, we can find a C curve S and a morphism ψ of $S - R$ to $\mathfrak{W}$. By passing to a ramified cover of S, we can assume ψ can be lifted to a map ψ' of $S - R$ to $\mathfrak{Y}$ so that ψ' induces a map of S to $S_{g,d}$. We may further assume that $\psi'(R)$ is a semistable point Q of $S_{g,d}$ and that if Q is strictly semistable, then the automorphism group of Q as a curve on G is infinite. By the results of [1], Q as an abstract curve is semistable and hence is either smooth or one of the X_n. By Corollary 3.2, Q has no continuous families of automorphisms as a curve in G. Thus $Q \in U$ and ψ' factors through U. Since φ is finite, ψ' factors through $\mathfrak{Y}$.

Now let T be a C scheme and let $G(T)$ consist of triples (Y, E, q) where Y is a semistable family of curves, of genus g over T, q: $Y \to \mathfrak{X} \times_C T$ satisfies (iii) above, and E is potentially stable on all the fibers of p: $Y \to T$. Then any element of $G(T)$ can be lifted locally to $F(T)$. We thus get a map of the functor G to $\mathfrak{W}$ which is bijective on closed points.

5. Let S be a smooth curve and let $R \in S$ be fixed. Let E' and E'' be bundles on S. We define a rational isomorphism φ of E' to E'' to be an isomorphism of E' with E'' over the generic point of S. There is a unique $r \in \mathbf{Z}$ such that φ induces a map

$$\varphi' \colon E'(rR) \to E''$$

which is defined (i.e., holomorphic) and nonzero at R. There is a unique s so that $(\operatorname{coker} \varphi')_R = \mathcal{O}_R/m^s$, where we define $m^0 = \mathcal{O}_R$. We say (r, s) is the *type* of φ with respect to E and E'.

Suppose E is a vector bundle on $X \times S$ and φ is an isomorphism of $(E_{P_1})_U$ with $(E_{P_2})_U$ where $U = S - R$. We can use φ as descent data to form a bundle

E_0 on $X_0 \times U$. Our aim in this section is to define a semistable family of curves X' and a bundle E' on X' extending $X_0 \times U$ and E_0 for certain types of φ. The resulting family will be called the geometric realization of φ.

First, if φ has type (r, s), then φ^{-1} has type $(-r - s, s)$. Indeed, notice that $Q_2 = \operatorname{coker}\varphi'$ is an $\mathcal{O}_R/m^s$ quotient of E_{P_2}. We will, in general, feel free to localize around R without comment. Thus we can assume that E_{P_2} is a direct sum $M_1 \oplus M_2$ where M_2 coincides with Q_2 over Spec $\mathcal{O}_R/m^s$. Thus

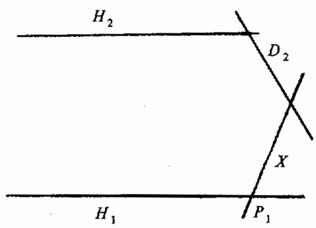
$$\operatorname{Ker}\left(E_{P_2} \to Q_2\right) = M_1 \oplus M_2(-sR).$$

Let $N_1 \oplus N_2$ be a direct sum decomposition of E_{P_1} which corresponds to $M_1 \oplus M_2(-sR)$ under the isomorphism of $E_{P_1}(rR)$ with $\operatorname{Ker}(E_{P_2}, Q_2)$. Thus we have isomorphisms

$$\varphi_1 \colon N_1(rR) \to M_1, \qquad \varphi_2 \colon N_2(rR) \to M_2(-sR).$$

Thus $M_i((-s - r)R)$ maps to N_i and this map is nonzero at R for $i = 2$. Further, $N_1/\varphi_1^{-1}(M_1((-r - s)R)) = \mathcal{O}_R/m^s$, so φ^{-1} has type $(-r - s, s)$.

Case (0, 1). We suppose φ has type (0, 1). This means that φ: $E_{P_1} \to \operatorname{Ker}(E_{P_2}, L_2)$, where L_2 is an R quotient of E_{P_2}. We let L_1 be the R quotient of E_{P_1} corresponding to the canonical quotient of $\operatorname{Ker}(E_{P_1}, L_2)$. On $X \times S$ blow up the point $P_2 \times R$ to obtain a surface $p\colon \mathcal{X}_1 \to X \times S$. Call the exceptional divisor D_2, and let the proper transform of $P_i \times S$ be H_i. Denote the proper transform of $X \times R$ by X.



Now let $E^{(2)} = p^*(E)$. Note that $p^*(L_2)$ is a D_2 quotient of rank one. Let $E' = \operatorname{Ker}(E^{(2)}, p^*(L_2))$. The map φ from E'_{H_2} to $\operatorname{Ker}(E_{P_2}, L_2)$ is an isomorphism. To obtain the geometric realization of φ, we glue H_1 to H_2 and use φ as an isomorphism of E'_{H_1} to E'_{H_2}. We have

(5.1.1) E'_{D_2} is standard of type 1.
(5.1.2) E'_X is the modification of $E_{X\times R}$ at L_2.
(5.1.3) At P_1, the standard quotient $\mathcal{O}$ of E'_{D_2} is glued to L_1, the $P_1 \times R$ quotient of $E'_{P_1} = E_{P_1}$.
(5.1.4) At P_2, the standard quotient $\mathcal{O}$ of E'_{D_2} is glued to the canonical quotient of E'_X from (5.1.2).

Indeed, (5.1.1) follows, since E_{D_2} is $\mathcal{O}_{D_2} \oplus \mathcal{O}_{D_2}$ and E'_{D_2} is $\mathcal{O}_{D_2} \oplus \mathcal{I}_{D_2}/\mathcal{I}^2_{D_2}$. Since $D_2^2 = -1$, $E'_{D_2} \cong \mathcal{O} \oplus \mathcal{O}(+1)$.

(5.1.2) and (5.1.4) follow from Remark 2.2. For (5.1.3), the standard quotient of E'_{D_2} is the canonical quotient, and L_1 is the quotient of E_{P_1} which is glued to the canonical quotient of $\mathrm{Ker}(E_{P_2}, L_2)$.

Case (1,0). If φ has type (1,0), we use the same surface $\mathcal{X}_1$ as in Case (0, 1). This time we have $\varphi\colon E_{P_1} \tilde{\to} E_{P_2}(-R)$. Letting $E^{(2)}$ be the pullback of E to $\mathcal{X}_1$, we define

$$E^{(3)} = \mathrm{Ker}\left(E^{(2)}, E^{(2)}_{D_2}\right).$$

We see that

(5.2.1) $E^{(3)}_X = E_X(-R)$.
(5.2.2) $E^{(3)}_{D_2}$ is $\mathcal{O}(1) \oplus \mathcal{O}(1)$.
(5.2.3) $E^{(3)}_{H_2} = E_{P_2}(-R)$.

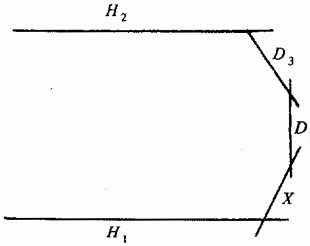
Case (1, 1). We have a quotient L_2 of $E_{P_2}(-R)$ and

$$\varphi\colon E_{P_1} \tilde{\to} \mathrm{Ker}\left(E_{P_2}(-R), L_2\right).$$

First, we proceed as in Case (1,0). This time, however, we have

$$\varphi\colon E^{(3)}_{H_1} \tilde{\to} \mathrm{Ker}\left(E^{(3)}_{H_2}, L_2\right).$$

So we blow up to $D_2 \cap H_2$ to obtain a new surface $\mathcal{X}_2$. Thus D_3 is the new exceptional divisor, and D_2 and H_2 are the proper transforms of D_2 and H_2 from $\mathcal{X}_1$. We let $E^{(4)}$ be the pullback of $E^{(3)}$ and let $E^{(5)} = \mathrm{Ker}(E^{(4)}, p^*(L_2))$, where $p\colon \mathcal{X}_2 \to \mathcal{X}_1$.



Now $E^{(5)}_{H_2}$ is isomorphic to $\mathrm{Ker}(E_{P_2}(-R), L_2)$ so φ extends to an isomorphism of $E^{(5)}_{H_1}$ to $E^{(5)}_{H_2}$. Thus we can obtain our geometric realization by using φ as glueing data for $E^{(5)}$. We claim:

(5.3.1) $E^{(5)}_{D_2}$ and $E^{(5)}_{D_3}$ are standard of degree 1. Further, their standard quotients do not coincide at $D_2 \cap D_3$.
(5.3.2) $E^{(5)}_X = E_{X \times R}(-R)$.

Indeed, $E_{D_3}^{(4)}$ is $\mathcal{O}_{D_3} \oplus \mathcal{O}_{D_3}$ which we modify at a quotient $\mathcal{O}_{D_3}$. Hence $E_{D_3}^{(5)}$ is $\mathcal{O}_{D_3} \oplus \mathcal{I}_{D_3}/\mathcal{I}_{D_3}^2 = \mathcal{O} \oplus \mathcal{O}(1)$. Further, the canonical quotient $E_{D_3}^{(5)}$ at $D_2 \cap D_3$ is just the standard quotient of $E_{D_3}^{(5)}$. On the other hand, $E_{D_2}^{(5)}$ is $\mathrm{Ker}(\mathcal{O}(1) \oplus \mathcal{O}(1), L_{D_2 \cap D_3})$. Thus $E_{D_2}^{(5)}$ is $\mathcal{O} \oplus \mathcal{O}(1)$, but the standard quotient does not coincide with the canonical quotient at $D_2 \cap D_3$. Thus the standard quotients of $E_{D_2}^{(5)}$ and $E_{D_3}^{(5)}$ do not coincide.

Case $(-1,2)$. If φ has type $(-1,2)$, then

$$\varphi\colon E_{P_1} \xrightarrow{\sim} \mathrm{Ker}\big(E_{P_2}, L_2\big)(+R).$$

Let L_1 be the quotient of E_{P_1} corresponding to the canonical quotient of $\mathrm{Ker}(E_{P_2}, L_2)(+R)$. Let $Q_i = L_i \otimes k_R$, where k_R is $\mathcal{O}_{S,R}/m$. We claim there is an isomorphism extending φ

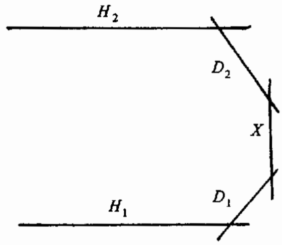
$$\psi\colon \mathrm{Ker}\big(E_{P_1}, Q_1\big) \xrightarrow{\sim} \mathrm{Ker}\big(E_{P_2}, Q_2\big).$$

Furthermore, ψ does not identify the canonical quotients of the two sides. We call Q_1 and Q_2 the quotients *induced* by φ. Indeed, using the notation developed earlier in this section, φ induces an isomorphism

$$\varphi\colon N_1(-R) \oplus N_2(-R) \xrightarrow{\sim} M_1 \oplus M_2(-2R).$$

Now $\mathrm{Ker}(E_{P_2}, Q_2)$ is $M_1 \oplus M_2(-R)$ and $\mathrm{Ker}(E_{P_1}, Q_1)$ is $N_1(-R) \oplus N_2$. Further, $(N_2)_R$ is the canonical quotient of $\mathrm{Ker}(E_{P_1}, Q_1)$ and $(M_1)_R$ is the canonical quotient of $\mathrm{Ker}(E_{P_2}, Q_2)$. Our claim follows.

Now form $\mathcal{X}_3$ by blowing up $\mathcal{X}_1$ at $P_1 = X \cap H_1$ to obtain a new exceptional divisor D_1 (see the diagram). Let $E^{(6)}$ be the pullback of E to $\mathcal{X}_3$ and let $E^{(7)}$ be the modification of $E^{(6)}$ at $p^*(Q_1) \oplus p^*(Q_2)$, where p denotes the projection of $\mathcal{X}_3$ to $X \times S$.



Then ψ is an isomorphism of $E_{H_1}^{(7)}$ to $E_{H_2}^{(7)}$ and so we can use ψ as descent data after identifying H_1 and H_2. We claim:

(5.4.1) $E_X^{(7)} \cong \mathrm{Ker}(E_X, Q_1 \oplus Q_2)$.
(5.4.2) $E_{D_i}^{(7)}$ is standard to type 1 and the ψ does not identify the standard quotients.

These can be verified using the above techniques.

Now suppose E is a family of stable bundles on X of degree $2\alpha + 1$ over S and let $\pi\colon X' \to S$ be the geometric realization of φ. In Case (0, 1), $X'_R = X_1$ and E'_R is of type II_1. Indeed, by Lemma 2.5, there is no semistabilizing quotient of $E_X^{(2)}$ which is identified with the standard D_2 quotient of $E_{D_2}^{(2)}$.

If φ is of type $(-1,0)$, then X'_R is X_1 and E'_R is of Type II_2. If φ is of type $(1,1)$, then X'_R is X_2, and E'_R is of Type III. Finally, if φ is of type $(-1,2)$, Lemma 2.5(ii) shows that X'_R is X_2, and E'_R is of Type III unless there is an invertible quotient L of degree $\alpha + 1$ of $E_{X\times R}$ coinciding with Q_1 and Q_2 over P_1 and P_2.

6. We continue with the same notation as §5. We suppose that φ has type $(-2,4)$. Let L_2 be the cokernel of the map from $E_{P_1}(-2R)$ to E_{P_2}, and let L_1 be the quotient of E_{P_1} corresponding to the canonical quotient of $\mathrm{Ker}(E_{P_2}, L_2)$. We suppose that there is an $X \times R$ quotient L of E which coincides with L_1 and L_2 at $R \times P_1$ and $R \times P_2$. Let $E' = \mathrm{Ker}(E, L)$. Notice first that φ has type $(-1,2)$ as a map of E'_{P_1} to E'_{P_2}. Indeed, φ induces an isomorphism

$$\varphi\colon N_1 \oplus N_2 \to M_1(+2R) \oplus M_2(-2R).$$

Further, $L_2 = M_2/M_2(-4R)$ and $L_1 = N_1/N_1(-4R)$. Thus

$$E'_{P_1} = N_1(-R) \oplus N_2, \qquad E'_{P_2} = M_1 \oplus M_2(-R).$$

Thus $E'_{P_1}(-R) = N_1(-2R) \oplus N_2(-R)$ and

$$\mathrm{Coker}\big(E'_{P_1}(-R), E'_{P_2}\big) = M_2(-R)/\varphi(N_2(-R)),$$

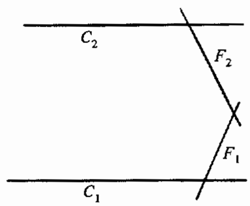
which is isomorphic to $\mathcal{O}_R/m^2$. So φ has type $(-1,2)$ as a map from E'_{P_1} to E'_{P_2}. Further, let Q'_1 and Q'_2 be the quotients of E'_{P_1} and E'_{P_2} induced by φ, and let L' be the canonical $X \times R$ quotient of $E' = \mathrm{Ker}(E, L)$. Then Q'_1 and Q'_2 do not coincide with L' over P_1 and P_2. Indeed, Q'_1 and Q'_2 correspond to $N_1(-R)$ and $M_2(-R)$ at P_1 and P_2 and L corresponds to N_2 and M_1 at P_1 and P_2. Let $S_2 = \mathrm{Spec}\, \mathcal{O}_{S,R}/m^2$.

Lemma 6.1. *Suppose there is a $X \times R$ quotient Q' of E' so that Q' coincides with Q'_1 and Q'_2 over P_1 and P_2 and so that $E'_{X\times R}$ is the direct sum of Q' and L'. Then there is an $X \times S_2$ quotient of E which coincides with L_1 and L_2 over $P_1 \times S_2$ and $P_2 \times S_2$ and with L on $X \times R$.*

Proof. Let $E'' = \mathrm{Ker}(E', Q')$. We claim E/E'' is the desired quotient. First, we must show E/E'' is an $X \times S_2$ quotient. Since the problem is local, we may assume there is an isomorphism $\psi\colon E \to M' \oplus M$ and that M coincides with L over $X \times R$. Then $E' = M' \oplus M(-D)$ where D is the divisor $X \times R$. M'_R is the canonical quotient of E', so the map from $M(-D)$ to Q' is an isomorphism. Locally, we can modify ψ so that the quotient $M(-D)$ of

$M' \oplus M(-D)$ coincides with Q'. Then $E'' = M' \oplus M(-2D)$, so $E/E'' = M/M(-2D)$. Now at P_1 and P_2, $(M_2(-R))_R$ and $(N_1(-R))_R$ coincide with the quotients induced by φ on E'_{P_1} and E'_{P_2}. Thus $E''_{P_1} = M_1 \oplus M_2(-R)$ and $E''_{P_2} = N_1(-2R) \oplus N_2$. Thus the E_{P_i}/E''_{P_i} coincides with L_i over S_2.

7. We wish to give a more global version of the construction of geometric realization of §5. Let T be a smooth variety and C_1, C_2, F_1 and F_2 are smooth divisors meeting transversally (see the diagram). We assume there are no triple intersections, and that C_2 meets only F_2, F_2 meets only C_2 and F_1, and C_1 meets only F_1. Suppose that E is a bundle on $X \times T$ and suppose we are given an isomoprhism



$$\varphi\colon E_{P_1} \xrightarrow{\sim} \operatorname{Ker}(E_{P_2}, L_2)(C_2 + F_2 - C_1), \tag{7.1}$$

where L_2 is an $F_1 + F_2$ quotient of E_{P_2}. We let L_1 be the quotient of E_{P_1} corresponding to the canonical $F_1 + F_2$ quotient corresponding to the right-hand side of (7.1).

Let S be a smooth curve on T and suppose $R \in S$. We suppose S is transversal at R to any C_i or F_i containing R. We let $\bar{E}$ be the restriction of E to $S \times X$, and $\bar{\varphi}$ the induced rational map from $\bar{E}_{P_1}$ to $\bar{E}_{P_2}$. In the following list, the first column represents the position of R and the second the type of $\bar{\varphi}$. Thus, if $R \in C_2 - F_2$, $\bar{\varphi}$ has type $(-1, 0)$:

$C_2 - F_2$	$(-1,0)$
$C_2 \cap F_2$	$(-2,1)$
$F_2 - C_2 - F_1$	$(-1,1)$
$F_2 \cap F_1$	$(-1,2)$
$F_1 - F_2 - C_1$	$(0,1)$
$F_1 \cap C_1$	$(1,1)$
$F_1 - D_1$	$(1,0)$

Our aim in this section is to construct a family of curves $q\colon Y \to T$ and a bundle F on Y so that for any S as above, the geometric realization of $\bar{\varphi}$ is just the pullback of Y and F to S. First, on $T \times X$ blow up $C_1 \times P_1$ and $C_2 \times P_2$ to obtain a variety W_1 mapping to $T \times X$. Let H_1 and H_2 be the proper transforms of $T \times P_1$ and $T \times P_2$, and let V_1 and V_2 be the exceptional divisors. Then H_i maps isomorphically to T, and $V_i \cap H_i$ maps to C_i. Now denoting the pullback of E by E again,

$$\varphi: E_{H_1} \to \operatorname{Ker}(E_{H_2}, L_2)(C_2 + F_2 - C_1).$$

Now on W_1, let

$$E' = \operatorname{Ker}(E, E_{V_1} \oplus E_{V_2}).$$

Then φ induces an isomorphism

$$\varphi: E'_{H_1} \to \operatorname{Ker}(E'_{H_2}, L'_2)(+F_2),$$

where L'_2 is the $F_1 + F_2$ quotient of E'_{H_2} given by $L'_2 = L_2(-C_2 + C_1)$. Now let L'_1 be the $F_1 + F_2$ quotient of E'_{H_1} corresponding to the canonical quotient of $\operatorname{Ker}(E'_{H_2}, L'_2)$. Then φ induces an isomorphism

$$\varphi_2: \operatorname{Ker}(E'_{H_1}, (L'_1)_{F_2}) \tilde{\to} \operatorname{Ker}(E'_{H_2}, (L'_2)_{F_1}).$$

Indeed, it is easy to check that φ_2 is an isomorphism over $F_1 - F_2$ and over $F_2 - F_1$. φ_2 is therefore an isomorphism since $\wedge^2\varphi_2$ does not vanish except possibly on $F_1 \cap F_2$ and hence does not vanish on $F_1 \cap F_2$.

Next in H_1, we have the smooth variety F'_2 corresponding to F_2 under the projection of H_1 to T. We can similarly define F'_1. Now blow up F'_2 and F'_1 in W_1 to obtain W_2. Let H_i denote the proper transform of H_i in W_2 and let G_1 and G_2 be the new exceptional divisors. We will let E' denote the pullback of E' to W_2 and M_1 and M_2 be the pullbacks $(L'_2)_{F_1}$ and $(L'_1)_{F_2}$. Then M_1 and M_2 are G_1 and G_2 quotients of E' on W_2. Let

$$E'' = \operatorname{Ker}(E', M_1 \oplus M_2).$$

Then φ induces an isomorphism of E''_{H_1} with E''_{H_2}. We glue H_1 to H_2 in W_2 to form our family of curves $q: Y \to T$ and use φ as descent data for E''. One checks that if $S \subseteq T$ is a curve as above, then the geometric realization of $\bar{\varphi}$ is just the pullback of Y and F.

8. Let S_0 be the moduli space of stable bundles of degree $2\alpha + 1$ on X and let E be the Poincaré bundle on $X \times S_0$. Our object is to construct the normalization of $\mathfrak{W}_P$ from S_0 and E, where $\mathfrak{W}$ is the variety introduced at the end of §4. We first consider

$$S_1 = \mathbf{P}(\mathbf{Hom}(E_{P_2}, E_{P_1}))$$

as a projective bundle over S_0. Let M_1 be the tautological bundle of π_1: $S_1 \to S_0$. We will denote the pullback of E to $S_1 \times X$ by $E^{(1)}$. Now on S_0, there is the usual exact sequence

$$\mathbf{Hom}\left(E^{(1)}_{P_2}, E^{(1)}_{P_1}\right) \to M_1 \to 0$$

and thus a map

$$\varphi_1 : E^{(1)}_{P_1} \to E^{(1)}_{P_2} \otimes M_1.$$

One checks locally that φ_1 is nowhere zero and that $\wedge^2 \varphi_1$ vanishes simply on a smooth divisor D which is a bundle of quadrics over S_0.

$$\mathcal{O}(D) \cong M_1^2 \otimes \wedge^2 E^{(1)}_{P_2} \otimes \left(\wedge^2 E^{(1)}_{P_1}\right)^{-1}. \tag{8.1.1}$$

There is a D quotient $L^{(1)}_2$ of $E^{(1)}_{P_2}$ so that

$$\varphi_1 : E^{(1)}_{P_1} \tilde{\to} \operatorname{Ker}\left(E^{(1)}_{P_2} \otimes M_1, L^{(1)}_2 \otimes M_1\right). \tag{8.1.2}$$

We let $L^{(1)}_1$ be the quotient of $E^{(1)}_{P_1}$ corresponding to the canonical quotient of the left-hand side of (8.1.2).

Consider

$$S_2 = \mathbf{P}(\mathcal{O} \oplus M_1)$$

as a projective bundle over S_1.

Let M_2 be the tautological line bundle for π_2: $S_2 \to S_1$. Let $E^{(2)}$ be the pullback of $E^{(1)}$ to $X \times S_2$, and $M^{(2)}_1$ be the pullback of M_1. We have the exact sequence on S_2

$$\mathcal{O} \oplus M^{(2)}_1 \to M_2 \to 0.$$

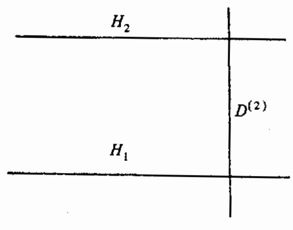
There are divisors H_1 and H_2 which are sections of π_2 so that the map from $\mathcal{O}$ to M_2 vanishes on H_2 and the map from $M^{(2)}_1$ vanishes on H_1. So we have isomorphisms $\mathcal{O} \tilde{\to} M_2(-H_2)$ and $M^{(2)}_1 \tilde{\to} M_2(-H_1)$. We obtain an isomorphism

$$\psi : \mathcal{O}(H_2 - H_1) \tilde{\to} M^{(2)}_1.$$

Thus we have a morphism

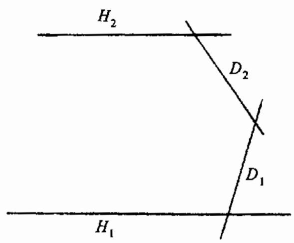
$$\varphi_2 : E^{(2)}_{P_1} \to E^{(2)}_{P_2}(H_2 - H_1).$$

We have the following schematic picture of S_2 where $D^{(2)} = \pi_2^{-1}(D)$:



Consider $U = S_2 - H_1 - H_2 - D^{(2)}$, and let $V_0 \subseteq \mathcal{W}_P$ be the open set of bundles of Type I_s. Note φ_2 is an isomorphism over U and we can use φ_2 as descent data to produce a bundle E' on $X_0 \times U$. E' is a family of bundle of Type I_s on X_0 and so we get a map ψ_0 of U to V_0. We claim ψ_0 is an isomorphism. Suppose T is a k scheme and E'' is a family of bundles on $T \times X_0$ of Type I_s. By pullback, we obtain a family of stable bundles $\tilde{E}$ on $T \times X$ together with an isomorphism $\tilde{\varphi}\colon \tilde{E}_{P_1} \to \tilde{E}_{P_2}$. There is a morphism F_0 of T to S_0 so that $\tilde{E}$ is the pullback of E up to tensoring by line bundles on T. From the universal property of $\mathbf{P}(\mathbf{Hom}(E_{P_2}, E_{P_1}))$, there is lifting P_1 of F_0 to S_1 and an isomorphism $h\colon F_1^*(M_1) \to \mathcal{O}$ so that $\tilde{\varphi}$ is the pullback of φ_1 when we use h to identify $\mathcal{O}$ with $F_1^*(M_1)$. But h determines the lifting F_2 of F_1 to $S_2 = \mathbf{P}(\mathcal{O} \oplus M_1)$ so that h is given as $\rho_1^{-1}\rho_2$, where ρ_2 is the map of $\mathcal{O}$ to $F_2^*(M_2)$ and ρ_1 is the isomorphism of $F_1^*(M_1) \to F_2^*(M_2)$. Thus $\tilde{\varphi}$ is just the pullback of φ_2. Hence, we obtain a map of T to U. Let $V \subseteq \mathcal{Y}$ be the open set consisting of curves of Type I_s. Then we have a map of V to U which is $\mathrm{SL}(W)$ invariant. Thus, we get a map ψ_1 of V_0 to U. We leave it to the reader to check that ψ_0 and ψ_1 are inverse maps.

It is impossible to extend ψ_0 to a map of S_2 to $\mathcal{W}_P$. So we must blow up S_2. First, blow up $H_2 \cap D^{(2)}$ to obtain S_3:



Here H_2 is the proper transform of H_2, D_2 is the new exceptional divisor, and D_1 is the proper transform of $D^{(2)}$. Thus, the total transform of $D^{(2)}$ is $D_1 + D_2$ and the total transform of H_2 is $H_2 + D_2$. Let $E^{(3)}$ be the pullback of $E^{(2)}$ to S_3 and let $L_2^{(3)}$ be the pullback of $L_2^{(1)}$. $L_2^{(3)}$ is a $D_1 + D_2$ quotient of $E_{P_2}^{(3)}$ and we have an isomorphism

$$\varphi_3\colon E_{P_1}^{(3)} \to \operatorname{Ker}\left(E_{P_2}^{(3)}, L_2^{(3)}\right)[H_2 + D_2 - H_1] \tag{8.1.3}$$

extending φ_2 on U. Thus, we are in the situation of §7. Let $L_1^{(3)}$ be the $D_1 + D_2$ quotient of $E_{P_1}^{(3)}$ corresponding to the canonical quotient of the right-hand side of (8.1.3). Let Z' be the subset of $D_1 \cap D_2$ consisting of points $s \in S_3$ so that E_3 has a quotient of degree $\alpha + 1$ which coincides with $L_i^{(3)}$ in $(E_s)_{P_i}$. From the discussion at the end of §§5 and 7, one sees that the family of curves and bundle obtained as the geometric realization of φ_3 is stable except over Z'.

9. Our purpose in this section is to understand Z'. Let J_d be the Picard variety of line bundles of degree d on X and let $\mathcal{L}_d$ be the Poincaré bundle on $X \times J_d$. We will denote the pullbacks of $\mathcal{L}_\alpha$ and $\mathcal{L}_{\alpha+1}$ to $X \times J_\alpha \times J_{\alpha+1}$ by $\mathcal{L}_\alpha$ and $\mathcal{L}_{\alpha+1}$. Let π_1 be the projection of $X \times J_\alpha \times J_{\alpha+1}$ onto $J_\alpha \times J_{\alpha+1}$. First consider the sheaf on $J_\alpha \times J_{\alpha+1}$

$$\mathcal{F}_1 = R^1\pi_{1*}\left(\mathcal{L}_{\alpha+1}^{-1} \otimes \mathcal{L}_\alpha\right). \tag{9.1.1}$$

First, note that $\mathcal{F}_1$ is locally free of rank $g - 1$ since if L is a line bundle of degree -1 on X, $h^1(L) = g - 1$. (The genus of X is $g - 1$.) Let $Z = \mathbf{P}(\mathcal{F}_1^{-1})$. We will construct an isomorphism of Z with Z'. Note dim $Z = 3g - 4$.

Again denoting the pullbacks of $\mathcal{L}_{\alpha+1}$ and $\mathcal{L}_\alpha$ to $Z \times X$ by $\mathcal{L}_\alpha$ and $\mathcal{L}_{\alpha+1}$ and letting π_2 be the projection of $Z \times X$ to Z, we have a canonical section s of $\mathfrak{M} \otimes R^1\pi_{2*}(\mathcal{L}_{\alpha+1}^{-1} \otimes \mathcal{L}_\alpha)$, where $\mathfrak{M}$ is the tautological bundle on Z over $J_\alpha \times J_{\alpha+1}$.

Now s defines an extension

$$0 \to \mathcal{L}_\alpha \otimes \mathfrak{M} \to \mathcal{E} \to \mathcal{L}_{\alpha+1} \to 0.$$

For each $z \in Z$, the corresponding extension

$$0 \to \left(\mathcal{L}_\alpha \otimes \mathfrak{M}\right)_z \to \mathcal{E}_z \to \left(\mathcal{L}_{\alpha+1}\right)_z \to 0$$

is nontrivial. We claim $\mathcal{E}_z$ is stable. Indeed, if Q_z is a destabilizing quotient of $\mathcal{E}_z$, the induced map f from $(\mathcal{L}_\alpha \otimes \mathfrak{M})_z$ to Q_z is nontrivial. But if f is zero at any point, deg $Q_z > \alpha$ so Q_z is not destabilizing. But the extension is trivial if f is an isomorphism.

Let T be a k-scheme. Let $G(T)$ be the set of extensions (modulo equivalence)

$$0 \to L_\alpha \to F \to L_{\alpha+1} \to 0 \tag{9.1.2}$$

on $X \times T$, where L_α and $L_{\alpha+1}$ have degree α and $\alpha + 1$ on $X \times \{t\}$ and where F is a family of stable bundles on X over T. Using the universal properties of the Picard group and $\mathbf{P}(\mathcal{F}_1^{-1})$, one sees that Z represents G.

We wish to define another functor G'. An element of $G'(T)$ will consist of a family F of stable bundles on $X \times T$ together with $T \times P_i$ quotients $\mathcal{L}_i$ of F (modulo equivalence). Recall the divisor $D \subseteq S_1$ from §8 and the D quotients $L_i^{(1)}$ of $E_{P_i}^{(1)}$. We claim D represents the functor G'. Indeed, given $(F, \mathcal{L}_1, \mathcal{L}_2) \in G'(T)$, we can locally on T find a φ: $F_{P_1} \to F_{P_2}$ so that coker $\varphi = \mathcal{L}_2$ and $F_{P_1}/\text{Ker}\,\varphi = \mathcal{L}_1$. Further, φ is uniquely determined up to a unit on T. Using the universal properties of S and S_1, one obtains a well-defined map from T to D so that $(F, \mathcal{L}_1, \mathcal{L}_2)$ is the pullback of $(E^{(1)}, L_1^{(1)}, L_2^{(1)})$ modulo isomorphism.

There is a map

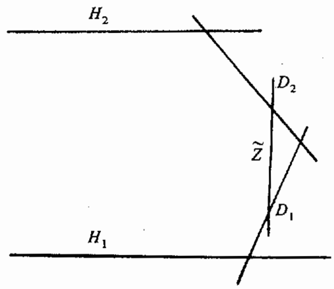
$$\Psi(T)\colon G(T) \to G'(T)$$

obtained by letting $\mathfrak{L}_i = (L_{\alpha+1})_{P_i}$. We claim Ψ is injective for any T. Indeed, suppose Q is a quotient of degree $\alpha+1$ of F in (9.1.2) which agrees with $L_{\alpha+1}$ at $P_i \times T$. Consider the map f of L_α to Q. This is actually a map of L_α to $Q(-P_1 - P_2)$. Since $\deg L_\alpha > \deg Q(-P_1 - P_2)$, f is zero. Thus $\Psi(T)$ is injective.

Proposition 9.2. *The induced map of Z to D is an embedding.*

Amplification 9.3. Let $T = \operatorname{Spec} k[\varepsilon]/(\varepsilon^2)$ and let R be the closed point of T. Let $(F, \mathfrak{L}_1, \mathfrak{L}_2) \in G'(T)$. Then the induced map of T to D factors through Z if and only if there is a T quotient $L_{\alpha+1}$ of F degree $\alpha+1$ coinciding with $\mathfrak{L}_1$ and $\mathfrak{L}_2$.

10. We retain the notation of §8. Using the map of Z to D and the isomorphism of $D_1 \cap D_2$ with D, we can identify Z with Z'. We form a new variety S_4 by blowing up $Z \subseteq S_3$. We will call the new exceptional divisor $\tilde{Z}$ and denote the proper transforms of the D_i by D_i.



Our main result of this section is

Proposition 10.1. *ψ_0 extends to a morphism of S_4 to $\mathfrak{W}_P$.*

Let $E^{(4)}$ be the pullback of $E^{(3)}$ and $L_2^{(4)}$ the pullback of $L_2^{(3)}$. Then $L_2^{(4)}$ is a $D_1 + D_2 + 2\tilde{Z}$ quotient of $E_{P_2}^{(4)}$ and there is an isomorphism

$$(10.1.1) \qquad \varphi_4\colon E_{P_1}^{(4)} \to \operatorname{Ker}\left(E_{P_2}^{(4)}, L_2^{(4)}\right)\left[H_2 + D_2 + \tilde{Z} - H_1\right].$$

$L_1^{(4)}$ will denote the quotient of $E_{P_1}^{(4)}$ corresponding to the canonical quotient of the right side of (10.1.1). Now let $\mathfrak{L}'_{\alpha+1}$ be the invertible quotient of $E_Z^{(3)}$ of degree $\alpha+1$ which coincides with $L_i^{(3)}$ over P_i. Let $\tilde{\mathfrak{L}}_{\alpha+1}$ be the pullback of $\mathfrak{L}_{\alpha+1}$ to $\tilde{Z}$. Further, let

$$(10.1.2) \qquad E' = \operatorname{Ker}\left(E^{(4)}, \tilde{\mathfrak{L}}_{\alpha+1}\right).$$

Now let $L_i' = L_i^{(4)} \otimes \mathcal{O}_{D_1+D_2}(-\tilde{Z})$. Note that L_i' is a $D_1 + D_2$ quotient of E'_{P_i}. We claim there is an isomorphism

$$\varphi' \colon E'_{P_1} \to \operatorname{Ker}(E'_{P_2}, L'_2)(H_2 + D_2 - H_1)$$

which coincides with φ_4 away from $\tilde{Z}$ and so that L'_1 coincides with the canonical quotient of $\operatorname{Ker}(E'_{P_2}, L'_2)$ under φ'. Our main object in this section is to show that the geometric realization of φ' is stable over all points of S_4.

To show that the isomorphism φ' exists, it suffices to work locally around a point of $\tilde{Z}$. Thus we may assume that

$$E^{(4)}_{P_1} = \mathcal{L}_1 \oplus \mathcal{M}_1, \qquad E^{(4)}_{P_2} = \mathcal{L}_2 \oplus \mathcal{M}_2,$$

where $\mathcal{L}_i$ coincides with $L_i^{(4)}$ over $D_1 + D_2 + 2\tilde{Z}$ and so that $\varphi_4 = \varphi_1 \oplus \varphi_2$ where φ_1 is an isomorphism of $\mathcal{L}_1$ with $\mathcal{M}_2(D_2 + \tilde{Z})$ and φ_2 is an isomorphism of $\mathcal{M}_1$ with $\mathcal{L}_2(-D_1 - \tilde{Z})$. Now $E'_{P_1} = \mathcal{L}_1(\tilde{Z}) \oplus \mathcal{M}_1$ and $E'_{P_2} = \mathcal{L}_2(-\tilde{Z}) \oplus \mathcal{M}_2$ and hence

$$\begin{aligned}\operatorname{Ker}(E'_{P_2}, L'_2)(D_2) &= \big(\mathcal{L}_2(-\tilde{Z} - D_1 - D_2) \oplus \mathcal{M}_2\big)(D_2) \\ &= \mathcal{L}_2\big(-\tilde{Z} - D_1\big) \oplus \mathcal{M}_2(D_2).\end{aligned}$$

Thus φ' exists, since $E'_{P_1} = \mathcal{L}_1(-\tilde{Z}) \oplus \mathcal{M}_1$.

Let Q be the canonical $\tilde{Z}$ quotient of E' from (10.1.2). Q_{P_1} and Q_{P_2} are just $(\mathcal{M}_{P_1})_Z$ and $(\mathcal{M}_{P_2})_Z$. So Q_{P_2} is actually a quotient of $\operatorname{Ker}(E'_P, L'_2)$. Further, Q_P is never glued to $Q_{P_2}(D_2)$ under φ' and Q_{P_i} never coincides with L'_i as a quotient of E'_{P_i}. Let N_1 be the quotient of $(E'_{P_1})_{\tilde{Z}}$ which corresponds to $Q_{P_2}(D_2)$ under φ'. Q_{P_2} is a quotient of $\operatorname{Ker}(E'_{P_2}, L'_2)$ which coincides with the canonical quotient of $\operatorname{Ker}(E'_{P_2}, L'_2)$ over $(D_1 + D_2) \cap \tilde{Z}$. Thus N_1 corresponds to L'_1 over $(D_1 + D_2) \cap \tilde{Z}$.

Since L'_1 corresponds to the canonical quotient of $\operatorname{Ker}(E'_{P_2}, L'_2)(D_2)$, one sees there is an isomorphism near $\tilde{Z}$

$$\psi' \colon E'_{P_2} \to \operatorname{Ker}(E'_{P_1}, L'_1)(+D_1)$$

which is the inverse of φ' except over $D_1 + D_2$. One defines N_2 analogously to N_1.

Now we have an exact sequence

$$0 \to M \to E'_{\tilde{Z} \times X} \to Q \to 0,$$

where M is a line bundle on $\tilde{Z} \times X$. Since the map φ' never identifies Q_{P_1} with $Q_{P_2}(+D_1)$, the map from M_{P_1} to $Q_{P_2}(+D_1)$ is an isomorphism. So we get a map $\theta_1 \in \operatorname{Hom}(Q_{P_2}, M_{P_1})$ which vanishes on D_1. Similarly, using ψ' one defines $\theta_2 \in \operatorname{Hom}(Q_{P_1}, M_{P_2})$ which vanishes on D_2.

Finally, let

$$E'' = \operatorname{Ker}\left(E'_{\tilde{Z}\times X}, N_1 \oplus N_2\right).$$

Note that Q is a quotient of E'' and that the kernel of $E'' \to Q$ is just $M(-P_1 - P_2)$. Thus we obtain an extension on $\tilde{Z} \times X$

$$0 \to M(-P_1 - P_2) \to E'' \to Q \to 0$$

and so a section

$$\theta_3 \in H^0\left(\tilde{Z}, R^1\pi_*\left(M \otimes Q^{-1}(-P_1 - P_2)\right)\right),$$

where π is the projection of $\tilde{Z} \times X$ to $\tilde{Z}$.

Now suppose $R \in \tilde{Z}$. We wish to examine the stability of the geometric realization (X_R, F_R) of φ' over R. First, suppose that $R \notin D_1 + D_2$. Then φ'_R: $(E'_P)_R \to (E'_{P_2})_R$ is an isomorphism and we have an exact sequence

$$0 \to M_R \to E'_R \to Q_R \to 0 \tag{10.1.3}$$

on $X \times R = X$. Now we have observed that Q_{P_1} and Q_{P_2} are not identified under φ'_R. Thus F_R on X_0 is of Type I_u, since $\deg Q_R = \alpha + 1$ and $\deg M_R = \alpha$. Suppose P is another point of $\tilde{Z} - D_1 - D_2$.

Lemma 10.2. *Suppose M_P and M_R are isomorphic and that Q_P and Q_R are isomorphic. Suppose that under these isomorphisms we have $\theta_i(R) = \lambda\theta_i(P)$ for $i = 1, 2, 3$ and $\lambda \in k^*$. Then F_R is isomorphic to F_P.*

Proof. We may first assume that $\lambda = 1$ by multiplying the isomorphism of M_P with M_R by a suitable constant. Since $\theta_3(R) = \theta_3(P)$, we can find an isomorphism ψ of the extensions:

$$0 \to M_R(-P_1 - P_2) \to E''_R \to Q_R \to 0$$

$$0 \to M_P(-P_1 - P_2) \to E''_P \to Q_P \to 0.$$

Now

$$E'_R = \operatorname{Ker}\left(E''_R, Q_{P_1} \oplus Q_{P_2}\right)(P_1 + P_2).$$

Thus there is an isomorphism of the extension (10.1.3) with the analogous extension over P so that $(N_i)_R$ corresponds to $(N_i)_P$. But $(N_1)_R$ is just the quotient of $(E'_{P_1})_R$ corresponding to $(Q_{P_2})_R$. Further, $\theta_1(R)$ just gives the map from $(M_{P_1})_R$ to $(Q_{P_2})_R$. Using the corresponding statements for $(N_2)_R$ and for P, our lemma follows from the following observation: Suppose V_1 and V_2 are two two-dimensional vector spaces given as extensions

$$0 \to U_i \to V_i \to W_i \to 0,$$

and that ψ and ψ' are isomorphisms of V_1 with V_2. Suppose $\psi(U_1) = \psi'(U_1)$, $\psi^{-1}(U_2) = (\psi')^{-1}(U_2)$ and the induced maps of U_1 to W_2 and U_2 to W_1 are equal. Then $\psi = \psi'$.

Next suppose $R \in (\tilde{Z} \cap D_1) - D_2$. The quotient $(L_2')_R$ does not coincide with $(Q_{P_2})_R$. Let $E'' = \mathrm{Ker}(E_R', L_R')$. Thus we have an exact sequence

$$0 \to M_R(-P_2) \to E'' \to Q_R \to 0.$$

Thus E'' is semistable and $(Q_R)_{P_2}$ is the canonical P_2 quotient of E''. On the other hand, $(N_1)_R \neq (Q_R)_{P_1}$. From (5.1.1)–(5.1.4), we see that F_R is on X_1, $(F_R)_X = E''$, $(Q_R)_{P_2}$ is glued to the standard quotient of $(F_R)_{R_1} = \mathcal{O} \oplus \mathcal{O}(1)$ and $(Q_R)_{P_1}$ is not glued to the standard quotient of $(F_R)_{R_1}$. Thus F_R is of Type II_1.

Finally, suppose $R \in \tilde{Z} \cap D_1 \cap D_2$. Let C be a curve passing through R transversal to $\tilde{Z}$, D_1 and D_2. We claim we cannot write

$$E_R' = M_R \oplus Q_R \tag{10.2.1}$$

where the quotient M_R coincides with $(L_i')_R$ over P_i. Now φ_C' has type $(-2, 4)$ as a map from $(E_C)_{P_1}$ to $(E_C)_{P_2}$. If (10.2.1) holds, Lemma 6.1 shows there is a $X \times C_2$ quotient of $E_C^{(4)}$ which coincides with $\tilde{\mathcal{L}}_{\alpha+1}$ on $X \times R$ and which coincides with $L_i^{(4)}$ over $P_i \times C_2$. But from Amplification 9.3, we see that the map from C_2 to S_1 factors through Z, i.e., the image of C in S_3 is tangent to Z. But C meets $\tilde{Z}$ transversally, and hence the image of C meets Z transversally. Thus our claim is established.

F_R is a bundle on X_2 and $(F_R)_X = \mathrm{Ker}(E_R', (L_1' \oplus L_2')_R)$. The L_i' coincide with N_i over R, and the N_i do not coincide with the Q_i. Lemma 2.6(ii) shows that $(F_R)_X$ is stable, and hence F_R is of Type III. Thus we have completed the proof of Proposition 10.1.

Lemma 10.3. *For any point $R \in \tilde{Z}$, at least one of the θ_i's is nonzero.*

Proof. θ_1 and θ_2 vanish only on D_1 and D_2 respectively so we may assume $R \in D_1 \cap D_2$. If $\theta_3(R) = 0$, we could find a splitting of

$$0 \to M(-P_1 - P_2)_R \to E_R'' \to Q_R \to 0.$$

But such a splitting would give a splitting of (10.2.1) so that M_R would coincide with $(N_i)_R$ over P_i. But $(N_i)_R = (L_i)_R$ and so we contradict our previous claim.

Let $\theta = (\theta_1, \theta_2, \theta_3)$ be the section of the bundle on $\tilde{Z}$ defined by

$$G = \mathbf{Hom}(Q_{P_2}, M_{P_1}) \oplus \mathbf{Hom}(Q_{P_1}, M_{P_2}) \oplus R^1\pi_*(M \otimes Q^{-1}(-P_1 - P_2)).$$

Now $Q = \tilde{\mathcal{L}}_\alpha \otimes \mathfrak{M}'$ and $M = \tilde{\mathcal{L}}_{\alpha+1} \otimes \mathcal{I}_{\tilde{Z}}/\mathcal{I}_{\tilde{Z}}^2$, where $\mathfrak{M}'$ is the pullback of the tautological bundle $\mathfrak{M}$ for $Z \to J_{\alpha+1} \times J_\alpha$. Consider the bundle $\mathcal{F}$ on $J_\alpha \times J_{\alpha+1}$ defined by

$$\begin{aligned}\mathcal{F} = \mathbf{Hom}((\mathcal{L}_\alpha)_{P_1}, (\mathcal{L}_{\alpha+1})_{P_2}) \oplus \mathbf{Hom}((\mathcal{L}_\alpha)_{P_2}, (\mathcal{L}_{\alpha+1})_{P_1})\\ \oplus R^1\pi_*(\mathcal{L}_{\alpha+1} \otimes \mathcal{L}_\alpha^{-1}(-P_1 - P_2)).\end{aligned}$$

Let $\mathcal{F}'$ be the pullback of $\mathcal{F}$ to Z. Then

$$G = p^*(\mathcal{F}') \otimes \mathcal{I}_{\tilde{Z}}/\mathcal{I}_{\tilde{Z}}^2 \otimes \mathfrak{M}^{-1},$$

where p is the projection from $\tilde{Z}$ to Z. Now using θ: $\mathcal{O} \to G$, we have a surjection

$$p^*\left((\mathcal{F}')^{-1} \otimes \mathfrak{M}\right) \to \mathcal{I}_{\tilde{Z}}/\mathcal{I}_{\tilde{Z}}^2 \to 0$$

and hence applying p_*, we see there is a surjective map

$$\eta\colon \mathfrak{M} \otimes (\mathcal{F}')^{-1} \to p_*\left(\mathcal{I}_{\tilde{Z}}/\mathcal{I}_{\tilde{Z}}^2\right) = \mathcal{I}_Z/\mathcal{I}_Z^2$$

on Z. Now by Riemann-Roch, the rank of $\mathcal{F}$ is $g + 1$. On the other hand, $\dim S_4 = 4 + \dim S_0 = 4g - 3$ and $\dim Z = 2g - 4$ so $\operatorname{rk} \mathcal{F} = \operatorname{codim}_{S_4} Z$. Thus η is an isomorphism since it is a surjective map between two bundles of the same rank. Since θ_1 vanishes on $D_1 \cap \tilde{Z}$, we see that $\mathbf{Hom}((\mathcal{L}_\alpha)_P, (\mathcal{L}_{\alpha+1}))_{P_2}^{-1} \otimes \mathfrak{M}$ corresponds to $(\mathcal{I}_{D_1})_Z \subseteq \mathcal{I}_Z/\mathcal{I}_Z^2$.

Thus $\tilde{Z} = \mathbf{P}(\mathcal{F}_1^{-1}) \times \mathbf{P}(\mathcal{F}^{-1})$, where $\mathcal{F}_1$ is defined by (9.1.1). Let p' be the projection of $\tilde{Z}$ to $\mathbf{P}(\mathcal{F}^{-1})$. Then Lemma 10.2 shows that ψ is constant on the fibers of p', at least over $\tilde{Z} - D_1 - D_2$. Now D_i maps to a divisor in $\mathbf{P}(\mathcal{F}^{-1})$, so ψ is constant on all the fibers, since the pullback of any very ample bundle on $\mathcal{U}_P$ is trivial on the fibers of p'. One sees that ψ restricted to $\tilde{Z}$ factors through $\mathbf{P}(\mathcal{F}^{-1})$.

We claim ψ maps $H_2 - D_2$ onto the set of bundles $U(\mathrm{II}_2)$ of Type II_2. Indeed, given a bundle $\tilde{F}_R$ on X_1 of Type II_2, we can find a bundle $\tilde{F}$ on the surface $\mathcal{X}_1$ of §5 and an isomorphism φ of $\tilde{F}_{H_1}$ with $\tilde{F}_{H_2}$ so that using φ as glueing data produces F_R over R. Now $\tilde{F}(+D_2)$ is the pullback of a family of stable bundles F_1 on $X \times S$ and we have

$$\varphi\colon (F_1)_{P_1} \xrightarrow{\sim} (F_1)_{P_2}(-R).$$

The geometric realization of φ over R is just F_R. The data of φ and F define a map of S to S_2 which passes transversally through $H_2 - D^{(2)}$ at R. Since the map of S_4 to S_2 is a local isomorphism at any point of $H_2 - D^{(2)}$, we get a map φ' of S to S_4 so that $\psi \circ \varphi'(R)$ is F_R. Similarly, $H_1 - D_1$ maps onto bundles of Type II_2.

We claim $H_1 \cap D_1$ maps onto the set $U(\mathrm{III})$ of bundles of Type III. Indeed, $U(\mathrm{III}) \subseteq \overline{U(\mathrm{II}_2)}$, so $U(\mathrm{III})$ is in the (closed) image of H_1. But no element of $H_1 - D_1$ maps to $U(\mathrm{III})$, so $H_1 \cap D_1$ maps onto $U(\mathrm{III})$. Next notice that D_1 maps onto $\overline{U(\mathrm{II}_1)}$. Indeed, $\psi(D_1)$ is strictly larger than $U(\mathrm{III})$ since no element of $D_1 - H_1 - H_2$ maps to $U(\mathrm{III})$. Since $U(\mathrm{III})$ is a divisor in $\overline{U(\mathrm{II}_1)}$, we see D_1 maps onto $\overline{U(\mathrm{II}_1)}$. Similarly, D_1 maps onto $\overline{U(\mathrm{II}_1)}$.

Lemma 10.5. *If D is a divisor on S_4, $D \neq \tilde{Z}$, then $\psi(D)$ is a divisor. Further, ψ restricted to $\tilde{Z}$ factors through* $\mathbf{P}(\mathcal{F}^{-1})$.

11. Suppose that $\tilde{S}$ and S' are smooth projective varieties and that φ: $\tilde{S} \to S'$ is a birational morphism. Suppose Z is smooth variety and that $\mathcal{F}$ is a bundle on Z. Suppose $\tilde{Z}$ is a division on $\tilde{S}$ and that we can identify $\tilde{Z}$ with $\mathbf{P}(\mathcal{F}^{-1})$ in such a way that $\mathcal{I}_{\tilde{Z}}/\mathcal{I}_{\tilde{Z}}^2$ is the tautological bundle on $\mathbf{P}(\mathcal{F}^{-1})$. [In our application, $\tilde{S} = S_4$, $S' = \tilde{\mathcal{W}}_P$, the normalization of $\mathcal{W}_P$.]

Lemma 11.1. *Suppose that if $D \subseteq \tilde{S}$ is any divisor with $D \neq \tilde{Z}$, then $\varphi(D)$ is a divisor. Suppose further that the induced map of $\tilde{Z}$ to S' factors through Z. Then the map of Z to S' is an inclusion, and $\tilde{S}$ is the blow up of S' at Z.*

Proof. Let ψ: $\varphi^*(\Omega_{S'}) \to \Omega_{\tilde{S}}$ be the induced map. Let U be the set of points over which ψ is an isomorphism. Then $D = \tilde{S} - U$ is a divisor given by $\det \psi = 0$. By Zariski's main theorem, we see that $\varphi^{-1}(\varphi(U)) = U$ and that for each $P \in D$, $\varphi^{-1}(\varphi(P))$ is connected and has dimension ≥ 1. Thus $D = \tilde{Z}$. Let π be projection of $\tilde{Z}$ to Z.

Now consider $\Omega_0 = \mathrm{Ker}(\Omega_{\tilde{S}}, \Omega_{\tilde{Z}/Z})$. The image of ψ is contained in Ω_0 since locally any function on S' is constant on the fibers of π. Now $\mathcal{O}(1) = \mathcal{I}_{\tilde{Z}}/\mathcal{I}_{\tilde{Z}}^2$. We first claim that $\det \Omega_0$ is trivial on the fibers of π. Indeed, we have the following exact sequences:

$$0 \to \mathcal{O}(1) \to (\Omega_{\tilde{S}})_{\tilde{Z}} \to \Omega_{\tilde{Z}} \to 0,$$

$$0 \to \pi^*\Omega_Z \to \Omega_{\tilde{Z}} \to \Omega_{\tilde{Z}/Z} \to 0,$$

$$0 \to \Omega_{\tilde{Z}/Z}(+1) \to \pi^*\mathcal{F}^{-1} \to \mathcal{O}(1) \to 0,$$

$$0 \to \Omega_{\tilde{Z}/Z}(+1) \to (\Omega_0)_{\tilde{Z}} \to (\Omega_{\tilde{S}})_{\tilde{Z}} \to \Omega_{\tilde{Z}/Z} \to 0.$$

So in $K(\tilde{Z})$, we have

$$[(\Omega_0)_{\tilde{Z}}] = [\pi^*\Omega_Z] + [\pi^*\mathcal{F}^{-1}].$$

Hence $\det \Omega_0$ is trivial on the fibers of π. Now the map from $\det \varphi^*\Omega_{S'}$ to $\det \Omega_0$ is an isomorphism outside $\tilde{Z}$, so

$$\det(\varphi^*\Omega_{S'}) = (\det \Omega_0)(-n\tilde{Z}).$$

But $\mathcal{O}(-n\tilde{Z})$ is nontrivial on the fibers of π if $n \neq 0$. So $n = 0$ and $\det \varphi^*(\Omega_{S'}) = \det \Omega_0$. Thus $\varphi^*(\Omega_{S'}) = \Omega_0$.

The map φ_Z from Z to S' has connected fibers since the map from $\tilde{Z}$ to S' has connected fibers. On the other hand, the map from $\varphi_Z^*(\Omega_{S'})$ to Ω_Z is surjective, since $\pi^*\Omega_Z$ is a quotient of Ω_0 and hence of $\varphi^*\Omega_S$. Thus φ_Z is an embedding. We identify Z with its image. There is an exact sequence on S'

$$0 \to \mathcal{I}_Z/\mathcal{I}_Z^2 \to (\Omega_{S'})_Z \to \Omega_Z \to 0.$$

Now $\operatorname{Ker}((\Omega_0)_{\tilde{Z}}, \varphi^*\Omega_Z)$ surjects to $\mathcal{I}_{\tilde{Z}}/\mathcal{I}_{\tilde{Z}}^2$, and hence $\varphi^*(\mathcal{I}_Z/\mathcal{I}_Z^2)$ maps surjectively to $\mathcal{I}_{\tilde{Z}}/\mathcal{I}_{\tilde{Z}}^2$. Thus, $\varphi^*(\mathcal{I}_Z) = \mathcal{I}_{\tilde{Z}}$. Let $\tilde{S}'$ be the blow up of S at Z and let E be the exceptional divisor. By the universal property of blowing up, S maps to $\tilde{S}'$ and $\tilde{Z}$ maps onto E and is therefore generically finite. As before, the map from the pullback of $\Omega_{\tilde{S}'}$ is an isomorphism, and hence $\tilde{S} = \tilde{S}'$.

12. Let S be a smooth projective variety and suppose $D_1, \cdots, D_n$ are smooth divisors which intersect transversally. Suppose $Z \subseteq D_1 \cap D_2$ is a smooth subvariety and $Z \cap D_k = \varnothing$ for $k > 2$. Further, suppose E and F are bundles on a smooth variety J and let $Z = \mathbf{P}(E)$. Let $\pi\colon Z \to J$ be the projection and let $\mathcal{O}(1)$ be the tautological bundle for π. Suppose $\mathcal{I}_Z/\mathcal{I}_Z^2 = \pi^*(F)(+1)$ and there are line bundles $L_i \subseteq F$ so that $\mathcal{I}_{D_i} \otimes \mathcal{O}_Z$ corresponds to $\pi^*(L_i)(+1)$.

Blow up Z on S to obtain $\tilde{S}$ with new exceptional divisor $\tilde{Z}$. Let $\tilde{D}_i$ be the proper transforms of D_i. Note that $\tilde{Z} = \mathbf{P}(E) \times_J \mathbf{P}(F)$. We will assume there is a map φ from $\tilde{S}$ to another smooth variety so that $\varphi_{\tilde{Z}}$ factors through $\mathbf{P}(F) = Z'$ and so that $\varphi(D)$ is a divisor on S' if $D \neq \tilde{Z}$. From §11, we see that $Z' \subseteq S'$ and that $\tilde{S}$ is the blow up of S' at Z'. Let D_i' be the images of $\tilde{D}_i$ in S'. First, notice that the D_i' intersect transversally. Indeed, $D_1' \cap Z'$ and $D_2' \cap Z'$ are the divisors on $Z' = \mathbf{P}(F)$ corresponding to the subbundles L_1 and L_2 of F.

Now let $k = \dim J$, let $m = \dim S$ and let $D = \Sigma D_i$. Our main goal in this section is

Proposition 12.1. *If $c_i(\Omega_S(\log D)) = 0$ for $i \geqslant m - k - 1$ and $c_i(\Omega_J) = 0$ for $i > 0$, then $c_i(\Omega_{S'}(\log D')) = 0$ for $i \geqslant m - k - 1$.*

The Chern classes may be taken in any convenient cohomology theory.

We begin with the following well-known consequence of Grothendieck's Riemann-Roch Theorem. Let X be a smooth projective variety and let H be a smooth divisor on X. Let E be a bundle of rank r on H and let $c(E) = \prod_r (1 + a_i)$ be the Chern polynomial of E on H, where the product is the usual formal device. Let h be the divisor class of $\mathcal{O}_H(H)$ on H and let i be the inclusion of H into X.

Lemma 12.2.

$$c(i_*E) = 1 + i_*\left(\frac{1 - \Pi((1 + a_i)/(1 + a_i - h))}{h}\right).$$

Proof. Let η_X^{-1} be the operator on $A(X)$ which multiplies a class of degree i by $(-1)^{i-1} \cdot (i-1)!$. Then

$$c(i_*E) = \exp\left(\eta_X^{-1} \cdot \operatorname{ch}(i_*E)\right).$$

From Grothendieck's Riemann-Roch formula for i, we know

$$\operatorname{ch}(i_*E) = i_*\left(\operatorname{ch} E \cdot \frac{1-e^{-h}}{h}\right).$$

Now in general

$$e^{i_*\alpha} = 1 + i_*((1 - e^{\alpha \cdot h})/h).$$

Hence

$$c(i_*E) = \exp\left(\eta_X^{-1}\left(i_*\left(\operatorname{ch} E \cdot \frac{1-e^{-h}}{h}\right)\right)\right) = \exp i_*\left(\frac{\eta_H^{-1}(\operatorname{ch} E \cdot (1-e^{-h}))}{h}\right)$$

$$= 1 + i_*\left(\frac{1 + \Pi((1+a_i)/(1+a_i-h))}{h}\right)$$

since $\operatorname{ch} E = \Sigma e^{a_i}$.

Suppose F is a bundle on X and that we have an exact sequence

$$0 \to F_1 \to F_H \to F_0 \to 0$$

of bundles on H. Let $F' = \operatorname{Ker}(F, F_0)$. Let r_1 be the rank of F_1 and r be the rank of F_0.

Lemma 12.3. *Suppose $c_i(F_1) = 0$ as a Chern class on H for $i \geqslant r_1 - k$, where k is some positive integer. Then $c_i(F) = c_i(F')$ on X for $i \geqslant r_1 + r - k$.*

Proof. On H, we write formally $c(F_0) = \Pi^r(1 + a_i)$. Then

$$c(F_0') = c(F_1) \cdot \Pi(1 + a_i - h),$$

since there is an exact sequence

$$0 \to F_0(-H) \to F_H' \to F_1 \to 0.$$

Now $c(F) = c(F') \cdot c(i_*F_0)$. Thus

$$c(F) - c(F') = c(F') \cdot \left(c(i_*(F_0)) - 1\right)$$

$$= c(F')i_*\left(\frac{1 - \Pi(1+a_i)/(1+a_i-h)}{h}\right)$$

$$= i_*\left(\frac{1}{h}\left\{\left(\prod^r(1+a_i-h) - \prod^r(1+a_i)\right)c(F_1)\right\}\right).$$

We see the expression in the braces has no components in degrees greater than $r + r_1 - k$.

Next we remark that if f is a rational function on P^k, and if the divisor of f is $H_1 - H_0$ where H_1 and H_0 are hyperplanes, then $d\log f$ is a nowhere zero section of $\Omega(\log(H_0 + H_1))$. Indeed, if $\{X_i\}$ are homogeneous coordinates with

$H_i = \{X_i = 0\}$ for $i = 0, 1$, then if $P \notin H_1 \cup H_2$, then f is a coordinate function and so $df \neq 0$ at P. On the other hand, if $P \in H_1$, then the residue of f around H_1 is nonzero.

We will next construct a canonical section s of $\Omega_{\tilde{Z}/J}(\log(\tilde{D}_1' + \tilde{D}_2'))$, where $\tilde{D}_i' = \tilde{D}_i \cap \tilde{Z}$. Now consider $z \in Z$ and let w be the projection of z in J. Now $(\mathcal{I}_{D_1} \otimes \mathcal{I}_{D_2}^{-1})_Z = \pi^*(L_1 \otimes L_2^{-1})$. Let (f_1, f_2) be a pair of functions defined in a neighborhood of z with f_i vanishing simply on D_i. We say (f_1, f_2) is good if $f_1 \otimes f_2^{-1}$ as a section of $(\mathcal{I}_{D_1} \otimes \mathcal{I}_{D_2}^{-1})_Z$ is the pullback of a section of $L_1 \otimes L_2^{-1}$. Note that if f_1' and f_2' are another good pair, then

$$\frac{z_1}{z_2} = \frac{z_1'}{z_2'} + f,$$

where f is a unit and f restricted to Z is the pullback of a function on J. Thus $d\log(z_1/z_2)$ gives a section s of $\Omega_{\tilde{Z}/J}(\log(\tilde{D}_1' + \tilde{D}_2'))$ which is independent of the good pair. The above remark shows that s projects to a nowhere zero section of $\Omega_{\tilde{Z}/Z}(\log(\tilde{D}_1' + \tilde{D}_2'))$. On the other hand, let $\bar{s}$ denote the projection of s in $\Omega_{\tilde{Z}/Z'}(\log(\tilde{D}_1' + \tilde{D}_2'))$. We claim $\bar{s} = 0$. Indeed, $\Omega_{\tilde{Z}/Z'}$ is a negative bundle on the fibers of $\pi'\colon \tilde{Z} \to Z'$, so $\bar{s}$ vanishes on $\tilde{Z} - (\pi')^{-1}(D_1' + D_2')$. Hence $\bar{s}$ vanishes. Now let $F \subseteq \Omega_{\tilde{Z}}(\log(\tilde{D}_1 + \tilde{D}_2))$ be the subsheaf generated locally by Ω_J and $d\log(z_1/z_2)$. Then we have an exact sequence

$$0 \to \pi^*\Omega_J \to F \to \mathcal{O}_{\tilde{Z}} \to 0.$$

Also, let $L \subseteq \Omega_{\tilde{Z}/Z}(\log(\tilde{D}_1' + \tilde{D}_2'))$ be the subline bundle generated by s, so $L \cong \mathcal{O}_{\tilde{Z}}$.

Consider on Q the sheaf defined by

$$0 \to \pi^*\Omega_S(\log D) \to \Omega_{\tilde{S}}(\log(\tilde{D} + \tilde{Z})) \to Q \to 0.$$

Lemma 12.4. $Q \cong \Omega_{\tilde{Z}/Z}(\log(\tilde{D}_1' + \tilde{D}_2'))/L$.

Proof. Consider the following diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & (\pi^*\Omega_S)(\log \tilde{D}) & \longrightarrow & \Omega_{\tilde{S}}(\log \tilde{D}) & \longrightarrow & \Omega_{\tilde{Z}/Z}(\log \tilde{D}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \psi & & \\
0 & \longrightarrow & \pi^*(\Omega_S(\log D)) & \longrightarrow & \Omega_{\tilde{S}}(\log(\tilde{D} + \tilde{Z})) & \longrightarrow & Q & \longrightarrow & 0 \\
 & & & \searrow^{\varphi} & \downarrow & & & & \\
 & & & & \mathcal{O}_{\tilde{Z}} & & & & \\
 & & & & \downarrow & & & & \\
 & & & & 0 & & & &
\end{array}$$

We will show that $\psi(L) = 0$ and the induced map $\bar{\psi}$ of $\Omega_{\tilde{Z}/Z}(\log \tilde{D})/L$ to Q is the required isomorphism. We first claim that φ is onto. Let $\tilde{z} \in \tilde{Z}$ and let z be projection of $\tilde{z}$ in Z. Let (f_1, f_2) be a good pair defined in a neighborhood of z. If $\tilde{z} \notin \tilde{D}_1$, then f_1 is a local equation for $\tilde{Z}$, so $d\log f_1$ generates $\mathcal{O}_{\tilde{Z}}$ at these points. If $\tilde{z} \in \tilde{D}_1$, then $f_1 = tt'$ where t and t' are local equations for $\tilde{D}_1$ and $\tilde{Z}$. Thus $d\log t$ is a section of $\Omega(\log \tilde{D})$, so since $d\log t' = d\log f_1 - d\log t$, we see that φ is onto. We further claim that any local section s of $\Omega_{\tilde{S}}(\log(\tilde{D}))$ which is also a section of $\pi^*(\Omega_S(\log D))$ maps to $L \subseteq \Omega_{\tilde{Z}/Z}(\log \tilde{D})$. Indeed, note that if $\varphi(h_1 d(\log f_1) - h_2 d(\log f_2)) = 0$, then we must have $h_1 = h_2$ on $\tilde{Z}$. Since all the sections of $\pi^*\Omega_S$ map to zero in $\Omega_{\tilde{Z}/Z}(\log \tilde{D})$, our claim follows. Finally, note that $\psi(L) = 0$ since $d\log(f_1/f_2) \in \pi^*(\Omega_S(\log D))$. It follows that $\bar{\psi}$ is an isomorphism.

Proof of Proposition 12.1. First note that

$$0 \to \mathcal{I}_{\tilde{Z}}/\mathcal{I}_{\tilde{Z}}^2 \to \left[\Omega_{\tilde{S}}(\log \tilde{D})\right]_{\tilde{Z}} \xrightarrow{\lambda} \left[\Omega_{\tilde{S}}(\log(\tilde{D} + \tilde{Z}))\right]_{\tilde{Z}} \to \mathcal{O} \to 0$$

is exact and that the image of λ is $\Omega_{\tilde{Z}}(\log(\tilde{D}_1 + \tilde{D}_2))$. Thus, F is a subbundle of $\Omega_{\tilde{S}}(\log(\tilde{D} + \tilde{Z}))_{\tilde{Z}}$. Further, F maps to zero in Q. Applying Lemma 12.3, and using the fact that $c_j(F) = 0$ for $j > 0$, we see that

$$0 = \pi^*\left(c_i\left(\Omega_S(\log D)\right)\right) = c_i\left(\pi^*\Omega_S(\log D)\right) = c_i\left(\Omega_{\tilde{S}}(\log(\tilde{D} + \tilde{Z}))\right)$$

for $i \geqslant m - k - 1$, since rk $F = k + 1$. Now F maps to zero in the quotient $\mathcal{O}_{\tilde{Z}}$ of $\Omega_{\tilde{S}}(\log(\tilde{D} + \tilde{Z}))$ since $d\log f_1/f_2$ does not have a pole on $\tilde{Z}$. Using a similar argument, we have

$$0 = c_i\left(\Omega_{\tilde{S}}(\tilde{D} + \tilde{Z})\right) = c_i\left(\Omega_{\tilde{S}}(\tilde{D})\right)$$

for $i \geqslant m - k - 1$. Now we also have an exact sequence

$$0 \to \pi'^*\left(\Omega_{S'}(\log D')\right) \to \Omega_S\left(\log \tilde{D}\right) \to \Omega_{\tilde{Z}/Z'}\left(\log\left(\tilde{D}\right)\right) \to 0.$$

Now $F \subseteq [\Omega_S(\log \tilde{D})]_{\tilde{Z}}$ maps to zero in $\Omega_{\tilde{Z}/Z'}(\log \tilde{D})$ by the above remark that $\bar{s} = 0$. Now applying Lemma 12.3 again, we see that

$$0 = c_i\left(\Omega_S(\log \tilde{D})\right) = c_i\left(\pi^*\left(\Omega_{S'}(\log D')\right)\right) = c_i\left(\Omega_{S'}(\log D')\right)$$

for $i \geqslant m - k - 1$.

13. Continuing with the notation of §10, let $Z' = \mathbf{P}(\mathcal{F}^{-1})$ and let $\tilde{\mathcal{W}}_P$ be the normalization of $\mathcal{W}_P$. The map ψ from S_4 to $\mathcal{W}_P$ factors $\tilde{\mathcal{W}}_P$, since S_4 is normal. Let $\tilde{\psi}$ be the induced map of S_4 to $\tilde{\mathcal{W}}_P$.

Theorem 13.1. *Z' is a subvariety of $\tilde{\mathcal{W}}_P$, and S_4 is the blow up of $\tilde{\mathcal{W}}_P$ at Z'. Further, if $D' \subseteq \tilde{\mathcal{W}}_P$ is the divisor which maps to the singular locus of $\mathcal{W}_P$, then $H_1 + H_2 + D_1 + D_2$ is the inverse image of D'.*

Corollary 13.2. *If $c_i(\Omega_{S_3}(\log(H_1 + H_2 + D_1 + D_2))) = 0$ for $i > 2g - 2$, then $c_i(\Omega_{\tilde{\mathcal{W}}_P}(\log D')) = 0$ for $i > 2g - 2$.*

Theorem 13.1 follows from §11 and Corollary 13.2 from §12.

Proposition 13.3. *$c_i(\Omega_{\tilde{\mathcal{W}}_P}(\log(D'))) = 0$ if $i > 2g - 2$ if $c_i(\Omega_{S_0}) = 0$ for $i >$* $2(g - 1) - 2$.

Proof. We will show first that

$$c_3\big(\Omega_{S_1/S_0}(\log D)\big) = 0.$$

We will work in the Chow ring modulo algebraic equivalence. First, notice that E_{P_1} and E_{P_2} are algebraically equivalent by letting P move from P_1 to P_2. Hence, we may formally write

$$c(E_{P_i}) = (1 + a_1)(1 + a_2).$$

Let $\alpha = a_1 - a_2$ and let ξ be the divisor class of M_1 on S_1. Thus $D = 2\xi$, since $\wedge^2 E_{P_1}$ and $\wedge^2 E_{P_2}$ are algebraically equivalent. Let $F = E_{P_2}^{-1} \otimes E_{P_1}$. Then we have the following exact sequences on S_1:

$$0 \to \Omega_{S_1/S} \to \pi_1^*(F)(-\xi) \to \mathcal{O} \to 0,$$

$$0 \to \Omega_{S_1/S} \to \Omega_{S_1/S}(\log D) \to \mathcal{O}_D \to 0,$$

$$0 \to \mathcal{O}(-D) \to \mathcal{O} \to \mathcal{O}_D \to 0.$$

Further, $\xi^2(\xi^2 - \alpha^2) = 0$ so

$$c(F(-\xi)) = (1 - \xi)^2\big((1 - \xi)^2 - \alpha^2\big).$$

Thus

$$\begin{aligned} c\big(\Omega_{S_1/S}(\log D)\big) &= \frac{(1 - \xi)^2\big((1 - \xi)^2 - \alpha^2\big)}{1 - 2\xi} \\ &= \frac{1}{1 - 2\xi}\Big[(1 - 2\xi)\big((1 - \xi)^2 - \alpha^2\big) + \xi^2(1 - 2\xi) + \xi^2(\xi^2 - \alpha^2)\Big] \\ &= \big((1 - \xi)^2 - \alpha^2\big) + \xi^2. \end{aligned}$$

Thus $c_3(\Omega_{S_1/S_0}(\log D)) = 0$.

We next note that $\Omega_{S_2/S_1}(\log(H_1 + H_2)) \cong \mathcal{O}$. Indeed, we can find a cover U_i of S_1 so that there are rational functions z_i on $\pi_2^{-1}(U_i)$ with a simple pole on H_1 and a simple zero on H_2. On $\pi_2^{-1}(U_i \cap U_j)$,

$$z_i = z_j \cdot f_{ij},$$

where the f_{ij} are units on $U_i \cap U_j$. Hence dz_i give a well-defined and nowhere zero section of $\Omega_{S_2/S_1}(\log(H_1 + H_2))$.

Now $\Omega_{S_2}(\log(D^{(2)} + H_1 + H_2))$ has a filtration whose successive quotients are $\Omega_{S_2/S_1}(\log(H_1 + H_2))$, $\pi_2^*(\Omega_{S_1/S_0}(\log D))$ and $\pi_2^*\pi_1^*(\Omega_{S_0})$. Hence

$$c_i\Big(\Omega_{S_2}\big(\log(D^{(2)} + H_1 + H_2)\big)\Big) = 0$$

if $i > 2g - 2$.

Now letting E be the divisor $D_1 + D_2 + H_1 + H_2$ on S_3, a local computation shows that

$$\pi^*\Omega_{S_2}(\log(D^{(2)} + H_1 + H_2)) = \Omega_{S_3}(\log E).$$

Applying Corollary 13.2 establishes Proposition 13.3.

Proof of Theorem 1.1. From Atyiah's classification of stable bundles on elliptic curves, the moduli space U_X is just X if X is an elliptic curve. Thus, Theorem 1.1 is true for $g = 1$.

Assume that the theorem is true for genus $g - 1$. There is a vector bundle $\Omega_{\mathcal{W}/C}(\log D')$ so that for $R \in C$, $R \neq P$, $(\Omega_{\mathcal{W}/C}(\log D'))_R$ is just the sheaf of one forms on the moduli space of stable vector bundles of degree $2\alpha + 1$ on $\mathcal{X}_R$. Now $\mathcal{W}_P$ is a deformation retract of $\mathcal{W}$, so to show $c_i(\Omega_{\mathcal{W}/C}(\log D')) = 0$, it suffices to show $c_i(\Omega_{\mathcal{W}_P}(\log D')) = 0$. Once this is done for $i > 2g - 2$, Theorem 1.1 will be established for $\mathcal{X}_R$. But once Theorem 1.1 is established for one smooth curve, it is established for any smooth curve.

Thus Theorem 1.1 will follow from

Lemma 13.4. *Let X be a projective scheme over* **C** *with normal crossings and let E be the bundle X. Suppose the X_j are the normalizations of the irreducible components of X. Then if E_j is the pullback of E to X_j, and $c_i(E_j) = 0$ for all j, then $c_i(E) = 0$.*

Proof. Let $2i = n$. Recall that there is functorial mixed Hodge structure on $H^n(X, \mathbf{C})$. Thus, there is a weight filtration

$$\cdots \subseteq W_{n-1} \subseteq W_n = H^n(X, \mathbf{Q})$$

and a Hodge filtration

$$0 \subseteq F^n \subseteq F^{n-1} \subseteq \cdots \subseteq F^0 = H^n(X, \mathbf{C}).$$

Now the natural map of $H^n(X, \mathbf{Q})$ to $H^n(\tilde{X}, \mathbf{Q})$ induces an injective map from W_n/W_{n-1} to $H^n(\tilde{X}, \mathbf{Q})$. Thus $c_i(E) \in W_{n-1}$, since $c_i(E)$ goes to zero in $H^n(\tilde{X}, \mathbf{Q})$.

On the other hand, we claim $c_i(E) \in F^i$. Indeed, let L be a very ample line bundle on X. Then we can find a map of X to a grassmannian G so that $E \otimes L^n$ is the pullback of the universal bundle for some n. We can find a map of X into $\mathbf{P}^N$ so that L^n is the pullback of $\mathcal{O}(1)$. Now let $\mathcal{E}$ be the universal bundle on G, and consider $\mathcal{E}(-1)$ on $G \times \mathbf{P}^N$. Then we can map X to $G \times \mathbf{P}^N$ so that E is the pullback of $\mathcal{E}(-1)$. Since $c_i(\mathcal{E}(-1))$ is in the ith level of the Hodge filtration and since the map of $H^n(G \times \mathbf{P}^n, \mathbf{Q})$ to $H^n(X, \mathbf{Q})$ is a morphism of Hodge structures, $c_i(E) \in F^i$.

Let F^k_{n-1} be the induced Hodge filtration on W_{n-1}/W_{n-2}. Then F^k_{n-1} is a pure Hodge structure of weight $n - 1$. In particular, $F^i_{n-1} \cap \overline{F^i_{n-1}} = 0$. Thus $c_i(E) \in W_{n-2}$. Continuing this line of reasoning, we see that $c_i(E) = 0$.

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